The Substitution Rule

MAT 126, Week 2, Wednesday class

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Substitution

Recall the Substitution Rule:

The Chain Rule states:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x).$$

By the FTC, taking integration on both side we get:

$$F(g(x)) = \int \frac{d}{dx}F(g(x)) = \int F'(g(x))g'(x)dx.$$

Substitution

The **Substitution Rule**: $F(g(x)) = \int F'(g(x))g'(x)dx$.

The key to the substitution rule is to find the part to be substitute, i.e. $u = g(x)$.

Recall the first Example of the Substitution rule:

Find: $\int 2x\sqrt{1+x^2}dx$. (Hint: use $u = 1 + x^2$.)

Substitution

Find: $\int 2x\sqrt{1+x^2}dx$. (Hint: use $u = 1 + x^2$.)

Solution:

- Step 1: Substitute $u = 1 + x^2$ (we choose this not because of the hint, but because of that this function is in the square root!)
- Step 2: Find $du = u'(x)dx = 2xdx$.
- Step 3: The original integration:

$$\int 2x\sqrt{1+x^2}dx = \int \sqrt{1+x^2}(2xdx) = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + C$$

- Step 4: Substitute $u = 1 + x^2$ back:

$$\int 2x\sqrt{1+x^2}dx = \frac{2}{3}(1+x^2)^{3/2} + C$$

Substitution

General Steps of the Substitution rule.

- Step 1: Find the correct substitution $u = u(x)$
- Step 2: Write $du = u'(x)dx$, so that $dx = \frac{du}{u'(x)}$
- Step 3: Plug $u = u(x)$ and $dx = \frac{du}{u'(x)}$ into the original integration. The terms involving x should all be cancelled.
  (After this step the integration should NOT involve any x, and you should expect that the new integration is much simpler to solve.)
- Step 4: Integrate the new integration (only involves u!)
- Step 5: Substitute the $u = u(x)$ back into the answer. (In the final answer there should NOT be any u).

Substitution

List of cases one should consider using the substitution $u = f(x)$.

- (1) Under the square root. $\sqrt{f(x)}$
- (2) Inside a power. $(f(x))^2$
- (3) Inside the trigonometry function $\sin(f(x))$
- ....

The list is not exhausted here!

Substitution

(5.5 E1) Find $\int x^3 \cos(x^4 + 2)dx$

Substitution

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Solution:

- Step 1: Substitute $u = x^4 + 1$.
- Step 2: Find $du = u'(x)dx = 4x^3 dx$. This implies $dx = \frac{du}{4x^3}$.
- Step 3: The original integration:

$$\int x^3 \cos(x^4 + 2)dx = \int x^3 \cos u \frac{du}{4x^3} = \frac{1}{4} \int \cos u du = \frac{1}{4} \sin u + C$$

- Step 4: Substitute $u = x^4 + 1$ back:

$$\int x^3 \cos(x^4 + 2)dx = \frac{1}{4} \sin(x^4 + 1) + C$$

Substitution

(5.5, E2) Find $\int \sqrt{2x+1}dx$

Substitution

(5.5, E2) Find $\int \sqrt{2x+1}dx$

Solution:

- Step 1: Substitute $u = 2x + 1$.
- Step 2: Find $du = u'(x)dx = 2dx$. This implies $dx = \frac{du}{2}$.
- Step 3: The original integration:

$$\int \sqrt{2x+1}dx = \int \sqrt{u}\frac{du}{2} = \frac{1}{2}\int \sqrt{u}du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}+C = \frac{1}{3}u^{3/2}+C$$

- Step 4: Substitute $u = 2x + 1$ back:

$$\int \sqrt{2x+1}dx = \frac{1}{3}(2x+1)^{3/2} + C$$

Substitution

(5.5 E3) Find $\int \frac{x}{\sqrt{1-4x^2}}dx$

Substitution

(5.5 E3) Find $\int \frac{x}{\sqrt{1-4x^2}} dx$

Solution:

- Step 1: Substitute $u = 1 - 4x^2$.
- Step 2: Find $du = u'(x)dx = -8xdx$. This implies $dx = -\frac{du}{8x}$.
- Step 3: The original integration:

$$\int \frac{x}{\sqrt{1-4x^2}} dx = -\int \frac{x}{\sqrt{u}} \frac{du}{8x} = -\frac{1}{8} \int \frac{du}{\sqrt{u}}$$

- Step 4: Integrate

$$-\frac{1}{8} \int \frac{du}{\sqrt{u}} = -\frac{1}{8} \int u^{-1/2} du = -\frac{1}{8} \frac{1}{-1/2+1} u^{1/2} + C = -\frac{1}{4}\sqrt{u} + C$$

- Step 5: Substitute $u = 1 - 4x^2$ back:

$$\int \frac{x}{\sqrt{1-4x^2}} dx = -\frac{1}{4}\sqrt{1-4x^2} + C$$

Substitution

(5.5 E4) Find $\int e^{5x} dx$

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(5.5 E4) Find $\int e^{5x} dx$

Solution:

- Step 1: Substitute $u = 5x$.
- Step 2: Find $du = u'(x)dx = 5dx$. This implies $dx = \frac{du}{5}$.
- Step 3: The original integration:

$$\int e^{5x} dx = \int e^{u} \frac{du}{5} = \frac{1}{5} \int e^{u} du = \frac{1}{5} e^{u} + C$$

- Step 4: Substitute $u = 5x$ back:

$$\int e^{5x} dx = \frac{1}{5} e^{5x} + C$$

Substitution

(5.5, 31) Find $\int x(2x+5)^8 dx$

Substitution

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Solution:

- Step 1: Substitute $u = 2x + 5$.
- Step 2: Find $du = u'(x)dx = 2dx$. This implies $dx = \frac{du}{2}$.
- Step 3: The original integration:

$$\int x(2x+5)^8 dx = \int x \cdot u^8 \frac{du}{2} = \int \frac{u-5}{2} \cdot u^8 \frac{du}{2}$$

- Step 3': Simplify the integration:

$$\int \frac{u-5}{2} \cdot u^8 \frac{du}{2} = \frac{1}{4} \int (u-5)u^8 du = \frac{1}{4} \int (u^9 - 5u^8) du$$

- Step 4: Integrate: $\frac{1}{4} \int (u^9 - 5u^8) du = \frac{1}{4}(\frac{u^{10}}{10} - \frac{5u^9}{9}) + C$
- Step 5: Substitute $u = 2x + 5$ back:

$$\int x(2x+5)^8 dx = \frac{1}{4}(\frac{(2x+5)^{10}}{10} - \frac{5(2x+5)^9}{9}) + C$$

Leading Principle of choosing the substitution:

Substitute $u = f(x)$ when there is an obvious $f'(x)$ in the integrant.

For example:

- If there is x^n inside some function (e.g. square root, sin...) and x^{n-1} outside, consider the substitution $u = x^n$, so that $du = (n-1)x^{n-1}dx$;
- If there is $\sin x$ and $\cos x$, consider the substitution $u = \sin x$, so that $du = \cos x dx$ (or vice versa);
- If there is $\ln x$ and $\frac{1}{x}$, consider the substitution $u = \ln x$, so that $du = \frac{1}{x}dx$
- If there is a $\tan x$ and $\sec^2 x$, consider the substitution $u = \tan x$, so that $du = \sec^2 x dx$
- ... (the list is not exhausted!)

Substitution using Trigonometry

(5.5 E5) Find $\int \tan x dx$

Substitution using Trigonometry

(5.5 E5) Find $\int \tan x dx$

Solution:

- Step 1: Note that $\tan x = \frac{\sin x}{\cos x}$. One can take $u = \cos x$. (Why not $u = \sin x$?)
- Step 2: Find $du = u'(x)dx = -\sin x dx$.
- Step 3: The original integration:

$$\int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\ln |u| + C$$

- Step 4: Substitute $u = \cos x$ back:

$$\int \tan x dx = -\ln |\cos x| + C$$

Substitution using Trigonometry

(5.5, 21) Find $\int \frac{\cos x}{\sin^2 x} dx$

Substitution using Trigonometry

(5.5, 21) Find $\int \frac{\cos x}{\sin^2 x} dx$

Solution:

- Step 1: Substitute $u = \sin x$.
- Step 2: Find $du = u'(x)dx = \cos x dx$.
- Step 3: The original integration:

$$\int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C$$

- Step 4: Substitute $u = \sin x$ back:

$$\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C$$

Substitution

(5.5 E8) Calculate $\int \frac{\ln x}{x} dx$

Substitution

Calculate $\int \frac{\ln x}{x} dx$

Solution:

- Step 1: Substitute $u = \ln x$.
- Step 2: Find $du = u'(x)dx = (1/x)dx$.
- Step 3: The original integration:

$$\int \frac{\ln x}{x} dx = \int u du = \frac{1}{2} u^2 + C$$

- Step 4: Substitute $u = \ln x$ back:

$$\int \frac{\ln x}{x} dx = \frac{1}{2} (\ln x)^2 + C$$

Substitution in a definite integral

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(5.5 E8) Calculate $\int_1^e \frac{\ln x}{x} dx$

Substitution in a definite integral

(5.5 E8) Calculate $\int_1^e \frac{\ln x}{x} dx$

Solution 1:

- By the previous problem we know that the antiderivative of $\frac{\ln x}{x}$ is $\frac{1}{2}(\ln x)^2 + C$.
- By evaluation theorem we have

$$\int_1^e \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2|_1^e = \frac{1}{2}(\ln e)^2 - \frac{1}{2}(\ln 1)^2 = \frac{1}{2}.$$

Substitution in a definite integral

(5.5 E8) Calculate $\int_1^e \frac{\ln x}{x} dx$

Solution 2:

- Step 1: Substitute $u = \ln x$. Then when x varies from 1 to e, u varies from $(\ln 1 =)0$ to $(\ln e =)1$.
- Step 2: $du = u'(x)dx = (1/x)dx$.
- Step 3: The original integration:

$$\int_1^e \frac{\ln x}{x} dx = \int_0^1 udu$$

- Step 4: Apply evaluation theorem to $\int_0^1 udu$ we have

$$\int_0^1 udu = \frac{1}{2}u^2|_0^1 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}.$$

Substitution in a definite integral

(5.5 E7) Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$

Substitution in a definite integral

(5.5 E7) Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$

Solution:

- Step 1: Substitute $u = 3 - 5x$. Then when x varies from 1 to 2, u varies from $(3 - 5 \cdot 1 =) - 2$ to $(3 - 5 \cdot 2 =) - 7$.
- Step 2: $du = u'(x)dx = -5dx$. Then $dx = -\frac{du}{5}$
- Step 3: The original integration:

$$\int_1^2 \frac{dx}{(3-5x)^2} = -\int_{-2}^{-7} \frac{du}{5u^2}$$

- Step 4: Apply evaluation theorem to $-\int_{-2}^{-7} \frac{du}{5u}$ we have

$$-\int_{-2}^{-7} \frac{du}{5u^2} = \frac{1}{5} \cdot (1/u)|_{-2}^{-7} = \frac{1}{5}\left(-\frac{1}{7} + \frac{1}{2}\right) = \frac{1}{14}$$

Symmetric functions

Symmetric functions:

- $f(x)$ is called an *even* function if for any number x, $f(x) = f(-x)$;
- $f(x)$ is called an *odd* function if for any number x, $f(x) = -f(-x)$;

Symmetric functions

Typical even functions:

- x^{2k} the even power of x (check: $(-x)^{2k} = (-1)^{2k}x^{2k} = x^{2k}$)
- any constant c is an even function (think of $c = c \cdot x^0$)
- $\cos x$ (check: $\cos(-x) = \cos x$)

Symmetric functions

Typical odd functions:

- x^{2k+1} the odd power of x (check: $(-x)^{2k+1} = (-1)^{2k+1}x^{2k+1} = -x^{2k+1}$)
- $\sin x$ (check: $\sin(-x) = -\sin x$)
- $\arctan x$ (check: $\arctan(-x) = -\arctan x$)

Symmetric functions

Properties of the symmetric functions:

- (1) If both $f(x)$ and $g(x)$ are even (or both are odd) functions, then $f(x) \cdot g(x)$ is an even function
- (2) If $f(x)$ is an even function and $g(x)$ is an odd function, then $f(x) \cdot g(x)$ is an odd function
- (Compare the two properties above with sums of integers.)
- (3) If $f(x)$ is an even (resp. odd) function, then $\frac{1}{f(x)}$ is an even (resp. odd) function.

Symmetric functions

Example:

Determine whether $\tan x$ is odd or even?

Symmetric functions

Determine whether $\tan x$ is odd or even.

Solution:

- Step 1: $\tan x$ is equal to $\frac{\sin x}{\cos x} = \sin x \cdot \frac{1}{\cos x}$.
- Step 2: since $\cos x$ is an even function, we know that $\frac{1}{\cos x}$ is an even function.
- Step 3: Since $\sin x$ is an odd function, the product $\sin x \cdot \frac{1}{\cos x}$ must be an odd function.
- So $\tan x$ is an odd function.

Symmetric functions

More Examples:

Even functions: (1) $1 + 3x^2 + 4x^4$; (2) $\sec x (= \frac{1}{\cos x})$; (3) $x^2 \cdot \cos x$; ...

Odd functions: (1) $x^5 + x^7$; (2) $\frac{sinx}{1+x^2}$; (3) $\cot x (= \frac{1}{\tan x})$; ...

Symmetric functions

One can use the substitution rule to show the following fact on the integral of an even function.

Q: If $f(x)$ is even ($f(x) = f(-x)$), show that $\int_{-a}^{a} f(x)dx = 2 \cdot \int_{0}^{a} f(x)dx$.

Symmetric functions

Q: If $f(x)$ is even ($f(x) = f(-x)$), show that $\int_{-a}^{a} f(x)dx = 2 \cdot \int_{0}^{a} f(x)dx$.

Solution:

- Step 1: Notice that $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$. This means we only need to show that $\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(x)dx$;
- Step 2: $\int_{-a}^{0} f(x)dx = -\int_{0}^{-a} f(x)dx$;
- Step 3: Use the substitute: $u = -x$. Then $du = -dx$. When x varies from 0 to $-a$, u varies from 0 to a.
- Step 4: We can rewrite $-\int_{0}^{-a} f(x)dx$ using u, as $-\int_{0}^{a} f(-u)(-du)$.
- Step 5: Since f is even, we have $f(-u) = f(u)$, so $-\int_{0}^{a} f(-u)(-du) = \int_{0}^{a} f(u)du$.

Symmetric functions

We can apply similar procedure to give the following property for the odd function:

Q:If $f(x)$ is odd ($f(-x) = -f(-x)$), Show that $\int_{-a}^{a} f(x)dx = 0$.

Symmetric functions

Q: If $f(x)$ is odd $(f(-x) = -f(-x))$, Show that $\int_{-a}^{a} f(x)dx = 0$.

Solution:

- Step 1: Notice that
  $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = -\int_{0}^{-a} f(x)dx + \int_{0}^{a} f(x)dx$.
- Step 2: This means we only need to show that
  $\int_{0}^{-a} f(x)dx = \int_{0}^{a} f(x)dx$;
- Step 3: Use the substitute: $u = -x$. Then $du = -dx$. When x varies from 0 to $-a$, u varies from 0 to a.
- Step 4: We can rewrite $\int_{0}^{-a} f(x)dx$ using u, as $\int_{0}^{a} f(-u)(-du)$.
- Step 5: Since f is odd, we have $f(-u) = -f(u)$, so
  $\int_{0}^{-a} f(x)dx = \int_{0}^{a} f(-u)(-du) = \int_{0}^{a} f(u)du$.

Symmetric functions

Application on the integral of an even function:

(5.5 E9) Find $\int_{-2}^{2}(x^6+1)dx$

(5.5 E9) Find $\int_{-2}^{2}(x^6+1)dx$

Solution:

- Step 1: Note that x^6+1 is an even function, we have $\int_{-2}^{2}(x^6+1)dx = 2\cdot\int_{0}^{2}(x^6+1)dx$
- Step 2: Compute $\int_{0}^{2}(x^6+1)dx$ using evaluation theorem:

$$\int_0^2 (x^6+1)dx = (\frac{1}{7}x^7+x)|_0^2 = \frac{1}{7}\cdot 2^7+2$$

- Step 3: So the original integral is equal to $2\cdot(\frac{1}{7}\cdot 2^7+2) = \frac{1}{7}\cdot 2^8+4$

Symmetric functions

Application on the integral of an odd function:

(5.5 E10) Find $\int_{-1}^{1} \frac{\tan x}{1+x^2+x^6} dx$

Symmetric functions

(5.5 E10) Find $\int_{-1}^{1} \frac{\tan x}{1+x^2+x^6} dx$

Solution:

- Since $\tan x$ is an odd function, and $1 + x^2 + x^6$ is an even function, their quotient must be an odd function.
- Since the interval of integration is symmetric (-1 to 1), so the integration must be equal to 0.

Other Substitution Questions

Other question that can be solved using substitution rule

(5.5, 67) If $\int_0^4 f(x)dx = 10$, find $\int_0^2 f(2x)dx$.

(5.5, 67) If $\int_0^4 f(x)dx = 10$, find $\int_0^2 f(2x)dx$.

Solution:

- Step 1: To find $\int_0^2 f(2x)dx$, consider the substitution $u = 2x$. Then when x changes from 0 to 2, u changes from 0 to 4.
- Step 2: $du = 2dx$, this means $dx = du/2$.
- Step 3: we have $\int_0^2 f(2x)dx = \int_0^4 f(u)\frac{du}{2} = \frac{1}{2}\int_0^4 f(u)du$
- Step 4: Since $\int_0^4 f(x)dx = 10$, we have $\frac{1}{2}\int_0^4 f(u)du = 5$.

Other Substitution Questions

(5.5, 68) If $\int_0^9 f(x)dx = 4$, find $\int_0^3 xf(x^2)dx$.

Other Substitution Questions

(5.5, 68) If $\int_0^9 f(x)dx = 4$, find $\int_0^3 xf(x^2)dx$.

Solution:

- Step 1: To find $\int_0^3 xf(x^2)dx$, consider the substitution $u = x^2$. Then when x changes from 0 to 3, u changes from 0 to 9.
- Step 2: $du = 2xdx$, this means $xdx = du/2$.
- Step 3: we have $\int_0^3 xf(x^2)dx = \int_0^9 f(u)\frac{du}{2} = \frac{1}{2}\int_0^9 f(u)du$
- Step 4: Since $\int_0^9 f(x)dx = 4$, we have $\frac{1}{2}\int_0^9 f(u)du = 2$.

Discussion

Discussion Problems

Use substitution to find the following integrals

- (5.5, 5) $\int \cos^3 x \sin x dx$ (use $u = \cos x$)
- (5.5, 9) $\int (3x-2)^{20} dx$
- (5.5, 13) $\int \frac{(\ln x)^2}{x} dx$
- (5.5, 45) $\int_0^1 x^2(1+x^3)^5 dx$
- (5.5, 50) $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} dx$
- (5.5, 51) $\int_1^2 x\sqrt{x-1} dx$