Integration by Parts

MAT 126, Week 2, Thursday class

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Integration by Parts

Recall that the substitution rule is a combination of the FTC and the chain rule. We can also combine the FTC and the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x).$$

Integration by Parts

Integrate the both sides of the product rule
$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$,, by the FTC, we have

$$f(x)g(x) = \int \frac{d}{dx}[f(x)g(x)]dx = \int f(x)g'(x)dx + \int f'(x)g(x)dx,$$

We can rearrange the terms and get

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx,$$

This is indeed the **Integration by Parts** formula.

Integration by Parts

We can write a perhaps more familiar form of the integration by parts formula by substituting $u = f(x)$ and $v = g(x)$, then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

gives

$$\int uv'dx = uv - \int vu'dx.$$

Remember that u, v are functions of x. Note that $du = u'(x)dx$, and $dv = v'(x)dx$, so we finally have

$$\int udv = uv - \int vdu$$

Integration by Parts

To apply the integration by parts formula:

$$\int u dv = uv - \int v du$$

The most important thing is to determine from the given integral which part should be your u, and which part should be your dv.

Let us see how to do this from the first example:

(5.6 E1) Find $\int x \sin x dx$

Integration by Parts

(5.6 E1) Find $\int x \sin x dx$

Solution:

- Step 1: In order to apply the integration by parts formula $\int udv = uv - \int vdu$, we let $u = x$, and let $dv = \sin x dx$.
- Step 2: We need to find out $du = dx$, $v = \int \sin x dx = -\cos x$
- Step 3: Plug in the formula $\int udv = uv - \int vdu$, we get

$$\int x \sin x dx = (x)(-\cos x) - \int (-\cos x) dx$$

- Step 4: Since $\int \cos x dx = \sin x$, we have the final answer as

$$-x \cos x + \sin x + C.$$

Summary of the Steps

In order to apply the integration by parts formula $\int u dv = uv - \int v du$, we need to know four data:

$$u, \qquad dv, \qquad du, \qquad v$$

- Step 1: Choose the suitable u and dv from the expression. Remember that dv should be of the form $v'(x)dx$.
- Step 2: From u and dv, we can find out

$$du = u'(x)dx \qquad \text{and} \qquad v = \int dv = \int v'(x)dx.$$

- Step 3: Plug in the formula $\int u dv = uv - \int v du$. The problem is now transformed into finding $\int v du$, which must be simpler than the original $\int u dv$.
- Step 4: Integrate $\int v du$. And get the final answer.

How to choose the correct u and dv?

One general principle:

Since we need to differentiate u to get du, and we need to integrate dv to get v,

we are choosing the u that is *easy to differentiate* and the dv that is *easy to integrate*!

For example: (5.6 E2) Evaluate $\int \ln x dx$

Integration by Parts

(5.6 E2) Evaluate $\int \ln x dx$

We have two choices:

- Choose $u(x) = 1$, and $dv = \ln x dx$;
- Or we choose $u(x) = \ln x$, and $dv = dx$.

(5.6 E2) Evaluate $\int \ln x dx$

If we go with the first choice: $u(x) = 1$, and $dv = \ln x dx$, then in order to find du and v, we need to calculate:

$$du = u'(x)dx = 0 \qquad dv = \int \ln x dx$$

This doesn't change the problem at all!

Integration by Parts

(5.6 E2) Evaluate $\int \ln x dx$

Solution:

- Step 1: Choose $u(x) = \ln x$, and $dv = dx$.
- Step 2: Find du and v:

$$du = u'(x)dx = \frac{1}{x}dx; \qquad v = \int dv = \int dx = x.$$

- Step 3: Plug in $\int udv = uv - \int vdu$, and get

$$\int \ln x dx = \ln x \cdot x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx$$

- Step 4: Since $\int dx = x$, we have

$$\int \ln x dx = x \ln x - x + C$$

Suggested choices on u and dv

Functions that are easy to differentiate (Usually will be set as u):

- x; x^2;... (So that by taking derivatives the power drops)
- $\ln x$
- $\arctan x$; $\arcsin x$

Functions that are easy to integrate (Usually will be set as dv):

- $\sin x dx$; $\cos x dx$
- $e^x dx$
- dx

Integration by Parts

(5.6, 3) Find $\int x \cos(5x) dx$

Integration by Parts

(5.6, 3) Find $\int x\cos(5x)dx$

Solution:

- Step 1: We can typically choose $u = x$, and $dv = \cos(5x)dx$
- Step 2: We have $du = dx$. To find

$$v = \int dv = \int \cos(5x)dx$$

  We will need to use substitution:

- Step 2': substitute $w = 5x$, then $dw = 5dx$. Then

$$v = \int \cos 5x dx = \int \cos w \frac{dw}{5} = \frac{1}{5}\sin w = \frac{1}{5}\sin(5x)$$

Integration by Parts

(5.6, 3) Find $\int x\cos(5x)dx$

Solution:

- Step 3: Plug in $\int udv = uv - \int vdu$, and get

$$\int x\cos(5x)dx = \frac{1}{5}\sin(5x)\cdot x - \frac{1}{5}\int \sin(5x)dx$$

- Step 4: We only need to integrate $\int \sin(5x)dx$, which we need to apply substitution ($w = 5x$), we will get $\int \sin(5x)dx = -\frac{1}{5}\cos x$.
- Step 5: The final answer is

$$\int x\cos(5x)dx = \frac{1}{5}x\sin(5x) - \frac{1}{5}\cdot(-\frac{1}{5}\cos x) = \frac{1}{5}x\sin(5x) + \frac{1}{25}\cos x + C$$

Integration by Parts

(5.6, 5) Find $\int re^{r/2}dr$

Integration by Parts

(5.6, 5) Find $\int re^{r/2}dr$

Solution:

- Step 1: We can typically choose $u = r$, and $dv = e^{r/2}dr$
- Step 2: We have $du = dr$. To find

$$v = \int dv = \int e^{r/2}dr$$

  We will need to use substitution:

- Step 2': substitute $w = r/2$, then $dw = \frac{1}{2}dr$. Then

$$v = \int e^w 2dw = 2e^w = 2e^{r/2}$$

Integration by Parts

(5.6, 5) Find $\int re^{r/2}dr$

Solution:

- Step 3: Plug in $\int udv = uv - \int vdu$, and get

$$\int re^{r/2}dr = r \cdot (2e^{r/2}) - \int 2e^{r/2}dr$$

- Step 4: We only need to integrate $\int e^{r/2}dr$, which we need to apply substitution ($w = r/2$), we will get $\int 2e^{r/2}dr = 2e^{r/2}$.
- Step 5: The final answer is

$$\int re^{r/2}dr = 2r \cdot e^{r/2} - 2\int e^{r/2}dr = 2r \cdot e^{r/2} - 4e^{r/2} + C$$

Integration by Parts

(5.6, 12) Find $\int \arcsin t dt$.

Integration by Parts

(5.6, 12) Find $\int \arcsin t dt$.

Solution:

- Step 1: Choose $u(t) = \arcsin t dt$, and $dv = dt$.
- Step 2: Find du and v:

$$du = u'(t)dt = \frac{1}{\sqrt{1-t^2}}dt; \qquad v = \int dv = \int dt = t.$$

- Step 3: Plug in $\int udv = uv - \int vdu$, and get

$$\int \arcsin t dt = t \arcsin t - \int \frac{t}{\sqrt{1-t^2}} dt$$

- Step 4: We now need to find $\int \frac{t}{\sqrt{1-t^2}} dt$. This is already solvable! (By substitution.)

Integration by Parts + Substitution

(5.6, 11) Find $\int \arcsin t dt$

Solution:

- Step 5: to find $\int \frac{t}{\sqrt{1-t^2}} dt$, let us substitute $w = 1 - t^2$.
- Step 6: From $w = 1 - t^2$ we get $dw = -2t dt$, so $t dt = -\frac{dw}{2}$
- Step 7: The integration becomes $\int \frac{t}{\sqrt{1-t^2}} dt = -\int \frac{1}{\sqrt{w}} \frac{dw}{2}$.
- Step 8: Integrate $-\int \frac{1}{\sqrt{w}} \frac{dw}{2}$ and get $-\frac{1}{2} \cdot 2w^{1/2} = -(1-t^2)^{1/2}$.
- Step 9: So the final answer is

$$\int \arcsin t dt = t \arcsin t + \sqrt{1-t^2} + C.$$

Integration by Parts

Another Example:

(5.6, 11) Find $\int \arctan t dt$

(5.6, 11) Find $\int \arctan t dt$

Solution:

- Step 1: Choose $u(t) = \arctan t dt$, and $dv = dt$.
- Step 2: Find du and v:

$$du = u'(t)dt = \frac{1}{1+t^2}dt; \qquad v = \int dv = \int dt = t.$$

- Step 3: Plug in $\int udv = uv - \int vdu$, and get

$$\int \arctan t dt = \arctan t \cdot t - \int t \cdot \frac{1}{1+t^2}dt = t \arctan t - \int \frac{t}{1+t^2}dt$$

- Step 4: We now need to find $\int \frac{t}{1+t^2}dt$. This is solvable by substitution.

Integration by Parts + Substitution

(5.6, 11) Find $\int \arctan t dt$

Solution:

- Step 5: to find $\int \frac{t}{1+t^2} dt$, let us substitute $w = 1 + t^2$.
- Step 6: From $w = 1 + t^2$ we get $dw = 2tdt$, so $tdt = \frac{dw}{2}$
- Step 7: The integration becomes $\int \frac{t}{1+t^2} dt = \int \frac{1}{w} \frac{dw}{2}$.
- Step 8: Integrate $\int \frac{1}{w} \frac{dw}{2}$ and get $\frac{1}{2} \ln |w| = \frac{1}{2} \ln |1 + t^2|$.
- Step 9: So the final answer is

$$\int \arctan t dt = t \arctan t - \frac{1}{2} \ln |1 + t^2| + C$$

Let us now do definite integral.

(5.6, E5) Calculate $\int_0^1 \arctan t dt$

(5.6, E5) Calculate $\int_0^1 \arctan t dt$

Solution:

- Step 1: From the last problem, we know that the antiderivative of $\arctan t$ is

$$t \arctan t - \frac{1}{2} \ln|1+t^2| + C$$

- Step 2: Use the evaluation theorem, we get

$$\begin{aligned}\int_0^1 \arctan t dt &= t \arctan t|_0^1 - \frac{1}{2} \ln|1+t^2||_0^1 \\ &= (1 \arctan 1 - 0 \arctan 0) - (\frac{1}{2} \ln|1+1^2| - \frac{1}{2} \ln|1+0^2|) \\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2\end{aligned}$$

Integration by Parts

From the previous examples we can see that the essential idea of the integration by parts formula

$$\int udv = uv - \int vdu$$

is to change the integral $\int udv$ into another integral $\int vdu$. The latter will be "easier" than the former to solve!

Note that "easier" usually means that we have done the same or similar problem before!

Integration by Parts

(5.6, 23) Find $\int_1^2 (\ln x)^2 dx$.

(5.6, 23) Find $\int_1^2 (\ln x)^2 dx$.

Solution:

- Step 1: We use $u = (\ln x)^2$, and $dv = dx$
- Step 2: Find du and v:

$$du = 2\ln x \cdot \frac{1}{x} dx \qquad v = \int dx = x$$

- Step 3: Plug in $\int u dv = uv - \int v du$, and find

$$\int_1^2 (\ln x)^2 dx = x(\ln x)^2 - \int x \cdot 2\ln x \cdot \frac{1}{x} dx$$
$$= x(\ln x)^2 - 2 \cdot \int \ln x dx$$

Our mission now is to integrate $\int \ln x dx$.

Integration by Parts (Twice!)

(5.6, 23) Find $\int_1^2 (\ln x)^2 dx$.

Solution:

- Step 4: We have done $\int \ln x dx$ in a previous example, it was done by integration by parts. (That means to solve this problem, we need to use integration by parts twice!)
- Step 5: We recall the solution: let $u = \ln x$ and $dv = dx$, so that $du = \frac{1}{x} dx$ and $v = x$.
- Step 6: By integration by parts, we have $\int \ln x dx = x \ln x - \int \frac{x}{x} dx = x \ln x - x$.

Integration by Parts (Twice!)

(5.6, 23) Find $\int_1^2 (\ln x)^2 dx$.

Solution:

- Step 7: Therefore the antiderivative to the original integrant is

$$\begin{aligned}\int (\ln x)^2 dx &= x(\ln x)^2 - 2 \cdot \int \ln x dx \\ &= x(\ln x)^2 - 2x \ln x + 2x + C \\ &= x\left((\ln x)^2 - 2\ln x + 2\right) + C\end{aligned}$$

- Step 8: The final answer:

$$\begin{aligned}\int_1^2 (\ln x)^2 dx &= x\left((\ln x)^2 - 2\ln x + 2\right)|_1^2 \\ &= 2\left((\ln 2)^2 - 2\ln 2 + 2\right) - 1\left((0)^2 - 2 \cdot 0 + 2\right) \\ &= 2(\ln 2)^2 - 4\ln 2 + 2\end{aligned}$$

Integration by Parts

(5.6 E3) Find $\int t^2 e^t \, dt$

Integration by Parts (Twice!)

(5.6 E3) Find $\int t^2 e^t dt$

Solution:

- Step 1: Use $u = t^2$, $dv = e^t dt$.
- Step 2: To find u and dv, we have

$$du = 2tdt, \qquad v = e^t$$

- Step 3: Plug in $\int udv = uv - \int vdu$, we have

$$\int t^2 e^t dt = t^2 e^t - \int e^t \cdot 2tdt = t^2 e^t - 2 \int te^t dt.$$

Now the new integral $\int te^t dt$ needs another integration by parts!

Integration by Parts (Twice!)

(5.6 E3) Find $\int t^2 e^t dt$

Solution:

- Step 4: To find $\int te^t dt$, we let $u = t$, $dv = e^t dt$.
- Step 5: We have $du = dt$, and $v = e^t$,
- Step 6: Plug in $\int udv = uv - \int vdu$, we have

$$\int te^t dt = te^t - \int e^t dt$$

Now we know the integral $\int e^t dt = e^t$.

- Step 7: The final answer:

$$\int t^2 e^t dt = t^2 e^t - 2 \int te^t dt = t^2 e^t - 2(te^t - e^t) + C$$

Integration by Parts

In the previous solution for $\int t^2 e^t dt$, can we use $u = e^t$ and $dv = t^2 dt$ instead?

Integration by Parts

The answer is NO! Let us see what will happen if we choose $u = e^t$ and $dv = t^2 dt$.

In this case, $du = e^t dt$ is all fine, but $v = \frac{1}{3}t^3$, the power of t is raised by 1!

The integration by parts gives

$$\int t^2 e^t dt = \frac{1}{3}t^3 e^t - \frac{1}{3}\int t^3 e^t dt$$

This is even more complicated!!

Integration by Parts

Through this example ($\int t^2 e^t dt$) we can see that, we need to use u to drop the power of t.

Namely, if there is a **positive** power of t in the integrant (t, t^2, t^3...), usually we will set $u = t$ (or $u = t^2$, $u = t^3$ ...) such that the differentiation $du = dt$ (or $du = 2tdt$, ...) will drop the power of t in the new integral $\int vdu$.

Remember that these are NOT strict rules!

Sometimes there is no good choice: Find $\int t \ln t dt$.

Integration by Parts

Find $\int t \ln t dt$.

- Step 1: Let $u = \ln t$, $dv = t dt$. (Why don't we use $u = t$ and $dv = \ln t dt$?)
- Step 2: $du = \frac{1}{t} dt$, $v = \frac{1}{2} t^2$.
- Step 3: Integration by parts yields:

$$\int t \ln t dt = \frac{1}{2} t^2 \ln t - \int \frac{1}{2} t^2 \frac{1}{t} dt = \frac{1}{2} t^2 \ln t - \frac{1}{2} \int t dt$$

- Step 4: We have $\int t dt = \frac{1}{2} t^2$. So the final answer is

$$\int t \ln t dt = \frac{1}{2} t^2 \ln t - \frac{1}{2} \cdot \frac{1}{2} t^2 = \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 + C$$

First make a substitution, then use integration by parts

(5.6, 29) Find $\int x \ln(1+x)dx$

Integration by Parts

(5.6, 29) Find $\int x \ln(1+x)dx$

Solution:

- Step 1: Since $(1+x)$ is inside the $\ln(1+x)$, we consider a substitution $t = 1 + x$.
- Step 2: By $t = 1 + x$, we have $x = t - 1$, and $dt = dx$.
- Step 3: The original integral is transformed into

$$\int x \ln(1+x)dx = \int (t-1) \ln t dt = \int t \ln t dt - \int \ln t dt.$$

We have dealt with both $\int \ln t dt$ and $\int t \ln t dt$ before!

(5.6, 29) Find $\int x \ln(1+x)dx$

Solution:

- Step 4: Recall that by letting $u = \ln t$ and $dv = dt$, we have $\int \ln t dt = t \ln t - t$.
- Step 5: Recall from the last example, by letting $u = \ln t$ and $dv = tdt$, we have $\int t \ln t dt = \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2$.
- Step 6: So the answer is

$$\begin{aligned} \int (1-t) \ln t dt &= -\int \ln t dt \int t \ln t dt \\ &= -t \ln t + t + \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 + C \\ &= t - \frac{1}{4}t^2 - \ln t + \frac{1}{2}t^2 \ln t + C \end{aligned}$$

(5.6, 29) Find $\int x \ln(1+x)dx$

Solution:

- Step 7: Plug the $t = 1 + x$ back, we have

$$\begin{aligned}\int x \ln(1+x)dx =&(1+x) - \frac{1}{4}(1+x)^2 \\ &- \ln(1+x) + \frac{1}{2}(1+x)^2 \ln(1+x) + C\end{aligned}$$

Integration by Parts

First make a substitution, then use integration by parts

(5.6 25) Find $\int \cos \sqrt{x} dx$

(5.6 25) Find $\int \cos \sqrt{x} dx$

Solution:

- Step 1: Let us first substitute $t = \sqrt{x}$. This is $x = t^2$.
- Step 2: By $x = t^2$ we have $dx = 2tdt$.
- Step 3: The original integral becomes $\int \cos t \cdot 2tdt = 2 \int t \cos tdt$. We now need to find $\int t \cos tdt$. This seems solvable because we have seen this before!

(5.6 25) Find $\int \cos\sqrt{x}dx$

Solution:

- Step 4: To find $\int t\cos tdt$ we need to use integration by parts. Let $u = t$, $dv = \cos tdt$.
- Step 5: We have $du = dt$, $v = \int \cos tdt = \sin t$.
- Step 6: Use integration by parts formula $\int udv = uv - \int vdu$, we have transformed the integral into

$$\int t\cos tdt = t\sin t - \int \sin tdt$$

Integration by Parts

(5.6 25) Find $\int \cos \sqrt{x} dx$

Solution:

- Step 7: We know that $\int \sin t dt = -\cos t$.

$$\int \cos \sqrt{x} dx = 2 \int t \cos t dt = t \sin t + \cos t + C$$

- Step 8: Substitute the $t = \sqrt{x}$ back:

$$\int \cos \sqrt{x} dx = \sqrt{x} \sin \sqrt{x} + \cos \sqrt{x} + C$$

Discussion

Discussion Problems

Use integration by parts to find the following integrals

- (5.6, 2) $\int \theta \cos \theta d\theta$
- (5.6, 15) $\int_0^\pi t \sin(3t) dt$
- (5.6, 17) $\int_1^2 \frac{\ln x}{x^2} dx$
- (5.6, 7) $\int x^2 \sin x dx$