The Fundamental Theorem of Calculus

MAT 126, Week 2, Monday class

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Review of the Riemann Sum

Review of the Riemann Sum

The Riemann Sum for a function f on the interval $[a, b]$:

- (Right) $R_n := \sum_{i=1}^{n} \Delta x \cdot f(x_i)$
- (Left) $L_n := \sum_{i=0}^{n-1} \Delta x \cdot f(x_i)$

where $\Delta x = \frac{b-a}{n}$, and $x_i = a + i \cdot \Delta x = a + i \cdot \frac{b-a}{n}$.

Review of the Riemann Sum

Definite Integral $\longleftrightarrow$ Limit of the Riemann Sum

$$\int_a^b f(x)dx \longleftrightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i)$$

- Given the definite integral, we can write down the limit of its Riemann Sum.
- Conversely, given the limit of a Riemann Sum, we can recover the corresponding definite integral.

Limit of a sum → Definite Integral

(Chap 5.2, 53) Express the limit as a definite integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$$

Limit of a sum $\rightarrow$ Definite Integral

Solution:

- **Step 0**: Note that $\frac{i^4}{n^5} = \frac{1}{n} \cdot \left(\frac{i}{n}\right)^4$, the original limit will be changed into

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^4 .$$

- **Step 1**: Compare the expression with the Riemann Sum formula $R_n = \Delta x \sum_{i=1}^{n} f(x_i)$ and conclude

$$\Delta x = \frac{1}{n}; \qquad f(x_i) = \left(\frac{i}{n}\right)^4$$

- **Step 2**: From $\Delta x = \frac{1}{n}$ we conclude that $a = 0, b = 1$.
- **Step 3**: From $a = 0, b = 1$ we can deduce $x_i = a + i \cdot \frac{b-a}{n} = \frac{i}{n}$.
- **Step 4**: From $x_i = \frac{i}{n}$ we see that

$$f(x_i) = \left(\frac{i}{n}\right)^4 = (x_i)^4.$$

  This implies $f(x) = x^4$.
- **Step 5**: We can now conclude that the definite integral is

$$\int_0^1 x^4 dx.$$

Summary of the Steps

- **Step 1**: Compare the expression with the Riemann Sum formula $R_n = \Delta x \sum_{i=1}^{n} f(x_i)$ and get Δx and $f(x_i)$ (Now the $f(x_i)$ is an expression WITHOUT the x_i).
- **Step 2**: From Δx we can get a, b (Usually we take $a = 0$, then $b = n \cdot \Delta x$);
- **Step 3**: From a, b we can get $x_i = a + i \cdot (b - a)/n$ (if we take $a = 0$, then $x_i = \frac{ib}{n}$);
- **Step 4**: From the x_i we can get the expression of $f(x_i)$ WITH the x_i, then we can get $f(x)$
- **Step 5**: We can now write down $\int_a^b f(x)dx$.

Limit of a sum → Definite Integral

(Chap 5.2, 54) Express the limit as a definite integral:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2}$$

Limit of a sum → Definite Integral

- **Step 1**: Compare the expression

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)^2}$$

with the Riemann Sum formula $R_n = \Delta x \sum_{i=1}^{n} f(x_i)$ and get

$$\Delta x = 1/n; \qquad f(x_i) = \frac{1}{1+(i/n)^2}$$

(Now the $f(x_i)$ is an expression WITHOUT the x_i).

- **Step 2**: From $\Delta x = 1/n$ we can get

$$a = 0, b = 1$$

Limit of a sum → Definite Integral

- **Step 3**: From $a = 0, b = 1$ we can get $x_i = 0 + i \cdot 1/n = i/n$;
- **Step 4**: From $x_i = i/n$ we can get

  $$f(x_i) = \frac{1}{1 + (i/n)^2} = \frac{1}{1 + x_i^2}$$

  Therefore we have

  $$f(x) = \frac{1}{1 + x^2}$$

- **Step 5**: Lastly, we have $\int_a^b f(x)dx = \int_0^1 \frac{1}{1+x^2} dx$.

The Fundamental Theorem of Calculus (FTC)

The FTC

The first example:

(5.4 E2) Let $g(x) = \int_1^x t^2 dt$, find a formula for $g(x)$ by evaluation theorem and calculate $g'(x)$.

The FTC

(5.4 E2) Let $g(x) = \int_1^x t^2 dt$, find a formula for $g(x)$ by evaluation theorem and calculate $g'(x)$.

Solution:

- The anti-der of t^2 is $\frac{1}{3}t^3$.
- By evaluation theorem: $g(x) = \frac{1}{3}t^3|_{t=1}^{t=x} = \frac{1}{3}x^3 - \frac{1}{3}$.
- Take derivative: $g'(x) = x^2$.

The FTC

Upshot: $g'(x) = x^2$ is the same function as the integrant (t^2)!

The FTC

General Case:

Theorem (The Fundamental Theorem of Calculus)
The function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$. Namely, $g'(x) = f(x)$.

Remark: As long as the lower limit is a constant (a), it doesn't matter what the number is!

The FTC

Let us check this fact with another example:

(5.4, 5) find the derivative of the function $g(x) = \int_0^x (1 + t^2)dt$.

The FTC

(5.4, 5) find the derivative of the function $g(x) = \int_0^x (1 + t^2)dt$.
Solution:

- The anti-der of $1 + t^2$ is $t + \frac{1}{3}t^3$.
- By evaluation theorem: $g(x) = (t + \frac{1}{3}t^3)|_{t=0}^{t=x} = (x + \frac{1}{3}x^3) - 0$.
- Take derivative: $g'(x) = (x + \frac{1}{3}x^3)' = x' + (\frac{1}{3}x^3)' = 1 + x^2$.

The FTC

Direct application of the FTC:

(5.4, E3) find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}dt$.

Solution: By the FTC, we have

$$g'(x) = \sqrt{1+x^2}.$$

The FTC

(5.4, 11) find the derivative of the function $g(x) = \int_x^{\pi} \sqrt{1 + \sec t}\, dt$.

The FTC

(5.4, 11) find the derivative of the function $g(x) = \int_x^\pi \sqrt{1 + \sec t}\,dt$.

Solution:

Firstly note that $\int_x^\pi \sqrt{1 + \sec t}\,dt = -\int_\pi^x \sqrt{1 + \sec t}\,dt$.

Then by the FTC, we have

$$g'(x) = -\sqrt{1 + \sec x}.$$

Differentiation and Integration as Inverse Processes

The two forms of the FTC:

Form 1 *(First differentiate, then integrate)*: $\int_a^b f(t)dt = F(b) - F(a)$, where F is any antiderivative of f. Namely:

$$\int_a^x F'(t)dt = F(x) - F(a).$$

(This is the Evaluation Theorem)

Form 2 *(First integrate, then differetiate)*: $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$. Namely:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

The FTC combined with the chain rule

(5.4 E5) Find $\frac{d}{dx} \int_1^{x^4} \sec t dt$.

Note that instead of x as the upper limit, we have x^4 as the upper limit.

The FTC combined with the chain rule

(5.4 E5) Find $\frac{d}{dx}\int_1^{x^4} \sec t dt$.

Solution: We need to use the Chain Rule:

- Step 1: Let $u = u(x) = x^4$ as the inner function; and let $y = y(u) = \int_1^u \sec t dt$ as the outer function.
- Step 2: The original question is now to find $\frac{d}{dx}y(u(x))$, which is a chain rule problem:

$$\frac{dy}{dx} = \frac{dy}{du}\Big|_{u=u(x)} \cdot \frac{du}{dx}.$$

The FTC combined with the chain rule

Solution: We need to use the Chain Rule:

- Step 3: Now we need to find $\frac{dy}{du}$ and $\frac{du}{dx}$:

$$\frac{du}{dx} = 4 \cdot x^3;$$

$$\frac{dy}{du} = \sec u.$$

- Step 4: Plug in the chain rule formula and find

$$\frac{dy}{dx} = \frac{dy}{du}\Big|_{u=u(x)} \cdot \frac{du}{dx} = \sec(x^4) \cdot 4 \cdot x^3 = 4x^3 \sec(x^4).$$

The FTC combined with the chain rule

(5.4, 13) Find the derivative of $h(x) = \int_2^{1/x} \arctan t dt$.

The FTC combined with the chain rule

Solution:

- Step 1: Let $u = 1/x$ be the inner function; let $y = \int_2^u \arctan t dt$ as the outer function.
- Step 2: Find the derivatives:

$$\frac{du}{dx} = -\frac{1}{x^2} \qquad \frac{dy}{du} = \arctan u.$$

- Step 3: Use the chain rule formula:

$$\frac{dy}{dx} = \frac{dy}{du}|_{u=u(x)} \cdot \frac{du}{dx} = \arctan(1/x) \cdot (-\frac{1}{x^2})$$

The FTC combined with the chain rule

(5.4, 17) Find the derivative of $g(x) = \int_{2x}^{3x} \frac{t^2-1}{t^2+1}\,dt$.

The FTC combined with the chain rule

(5.4, 17) Find the derivative of $g(x) = \int_{2x}^{3x} \frac{t^2-1}{t^2+1} dt$.

Solution:

- Step 1: Need to firstly break the integration into two:

$$\begin{aligned}
\int_{2x}^{3x} \frac{t^2-1}{t^2+1} dt &= \int_{0}^{3x} \frac{t^2-1}{t^2+1} dt + \int_{2x}^{0} \frac{t^2-1}{t^2+1} dt \\
&= \int_{0}^{3x} \frac{t^2-1}{t^2+1} dt - \int_{0}^{2x} \frac{t^2-1}{t^2+1} dt \\
&=: g_1(x) - g_2(x)
\end{aligned}$$

- Step 2: Apply the FTC + chain Rule to $g_1(x)$ and $g_2(x)$ separately.

The FTC combined with the chain rule

For $g_1(x)$:

- Let $u = 3x$ be the inner function; let $y = \int_0^u \frac{t^2-1}{t^2+1} dt$ as the outer function.
- Find the derivatives:

$$\frac{du}{dx} = 3 \qquad \frac{dy}{du} = \frac{u^2-1}{u^2+1}.$$

- Step 4: Use the chain rule formula:

$$g_1'(x) = \frac{dy}{dx} = \frac{dy}{du}\Big|_{u=u(x)} \cdot \frac{du}{dx} = 3 \cdot \frac{(3x)^2-1}{(3x)^2+1}$$

The FTC combined with the chain rule

Similarly for g_2 we have

$$g_2'(x) = 2 \cdot \frac{(2x)^2 - 1}{(2x)^2 + 1}$$

.

Together we have:

$$g'(x) = g_1'(x) - g_2'(x) = 3 \cdot \frac{(3x)^2 - 1}{(3x)^2 + 1} - 2 \cdot \frac{(2x)^2 - 1}{(2x)^2 + 1}.$$

The FTC combined with the chain rule

(5.4, 22) If $f(x) = \int_0^{\sin x} \sqrt{1+t^2}dt$ and $g(y) = \int_3^y f(x)dx$, find $g''(\pi/6)$

The FTC combined with the chain rule

(5.4, 22) If $f(x) = \int_0^{\sin x} \sqrt{1+t^2}dt$ and $g(y) = \int_3^y f(x)dx$, find $g''(\pi/6)$

Solution:

- Step 1: To find $g''(\pi/6)$ we need first find $g''(x)$, then plug in $x = \pi/6$.
- Step 2: First derivative: $g'(x) = f(x)$ (1st time of the FTC)
- Step 3: Second derivative: $g''(x) = f'(x) = \frac{d}{dx} \int_0^{\sin x} \sqrt{1+t^2}dt$. This is the 2nd time of the FTC, we need to use the chain rule in this step.

The FTC combined with the chain rule

- Step 4: To find $\frac{d}{dx}\int_0^{\sin x}\sqrt{1+t^2}dt$, let $u=\sin x$ be the inner function; let $y=\int_0^u\sqrt{1+t^2}dt$ as the outer function.
- Step 5: Find the derivatives:

$$\frac{du}{dx}=\cos x \qquad \frac{dy}{du}=\sqrt{1+u^2}$$

- Step 6: Use the chain rule formula:

$$g''(x)=f'(x)=\frac{dy}{dx}=\frac{dy}{du}|_{u=u(x)}\cdot\frac{du}{dx}=\sqrt{1+(\sin x)^2}\cdot\cos x.$$

- Step 7: Plug in:
  $g''(\pi/6)=\sqrt{1+(\sin(\pi/6))^2}\cdot\cos(\pi/6)=\sqrt{1+(\frac{1}{2})^2}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{15}}{4}$.

Substitution

Substitution

Let us now see another version of the *FTC + Chain Rule* story:

The Chain Rule states:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x).$$

By the FTC, taking integration on both side we get:

$$F(g(x)) = \int \frac{d}{dx}F(g(x)) = \int F'(g(x))g'(x)dx.$$

This is called the **Substitution Rule**.

Substitution

The **Substitution Rule**: $F(g(x)) = \int F'(g(x))g'(x)dx$.

The key to the substitution rule is to find the part to be substitute, i.e. $u = g(x)$.

First Example of the Substitution rule:

Find: $\int 2x\sqrt{1+x^2}dx$. (Hint: use $u = 1 + x^2$.)

Substitution

Find: $\int 2x\sqrt{1+x^2}dx$. (Hint: use $u = 1 + x^2$.)

Solution:

- Step 1: Substitute $u = 1 + x^2$ (we choose this not because of the hint, but because of that this function is in the square root!)
- Step 2: Find $du = u'(x)dx = 2xdx$.
- Step 3: The original integration:

$$\int 2x\sqrt{1+x^2}dx = \int \sqrt{1+x^2}(2xdx) = \int \sqrt{u}du = \frac{2}{3}u^{3/2}$$

- Step 4: Substitute $u = 1 + x^2$ back:

$$\int 2x\sqrt{1+x^2}dx = \frac{2}{3}(1+x^2)^{3/2}.$$

Discussion

Discussion Problems

Use the Fundamental Theorem of Calculus to find the derivatives of the functions

- (5.4, 7) $g(x) = \int_1^x \frac{1}{t^3+1} dt$
- (5.4, 9) $g(x) = \int_2^x t^2 \sin t dt$
- (5.4 14) $h(x) = \int_0^{x^2} \sqrt{1+r^3} dr$
- (5.4 15) $f(x) = \int_0^{\tan x} \sqrt{t+\sqrt{t}} dt$

Evaluate the integral by using the given substitute

- (5.5, 1) $\int e^{-x} dx$, (use $u = -x$).