MAT 331 Fall 2023 Project
Points of Increase

This project is based on the paper "Points of increase for Brownian motion" by Yuval Peres. A copy of this paper is available on the MAT 331 website. Consider a random walk on the integers that moves left or right at each step with equal probability. Let X_k denote the position of the random walk at step k (assume the walk starts at $X_0 = 0$). Suppose the walk has N steps. We say n, for $1 \leq n \leq N$ is a point of increase if $X_k \leq X_n$ for all $0 \leq k \leq n$ and $X_k \geq X_n$ for all $n \geq k \geq N$.

(1) Write a MATLAB script that takes in a random walk (a list of real numbers) and checks whether this random walk has a point of increase or not. Test it on a random walk of length $N = 1000$ created by summing a sequence of random -1's and $+1$'s.

(2) Choose a large M, say $M = 10,000$, and repeat the experiment M times. How many times does the random walk have a point of increase?

(3) Peres' paper gives an upper and lower bound for the probability that there is a point of increase in terms of quantities

$$p_n = \text{ probability that } X_k \geq 0 \text{ for all } 0 \leq k \leq n.$$

The precise values for these are given by

$$p_n = \binom{n}{n/2} 2^{-n}, \ n \text{ even},$$

$$p_n = \binom{n}{(n-1)/2} 2^{-n}, \ n \text{ odd},$$

which you may use without proof (a proof is given in Section 6.11 of the book by Bishop and Peres, *Fractals in probability and analysis* which can be found on the homepage). Write code to compute these values. Note that $p_0 = 1$, but that MATLAB arrays start with index 1, so you have to work around this somehow (there are several ways, e.g., make p a function instead of an array). For values of n up to around 1000, you can use the `nchoosek` command to compute the binomial coefficients, but you will get messages about lack of accuracy (these won't effect your answer much). You can suppress warning messages in your output using the `warning('off')` command. Alternatively, using the formulas above, it is not hard to derive a formula for p_{n+2} in terms of p_n, and using this will give a better result for much larger values of n.

(4) Using the formulas (3) and (5) in Peres' paper, compute the lower and upper bound for the case you did in the experiment.

(5) How does your experimental result compare to the theoretical predictions?