MAT 331 Fall 2023 Project
The box-counting dimension of the Weierstrass function

The Weierstrass function on $[0,1]$ is defined by

$$f_{b,\alpha}(x) = \sum_{n=0}^{\infty} b^{-\alpha n} \sin(2\pi b^n x).$$

where b is an integer greater or equal to 2 and $0 < \alpha < 1$. It is a famous example of a continuous function that is nowhere differentiable. You may have encountered it in MAT319/320. The graph of this function is also an example of a "fractal" set. This project concerns numerically estimating the box-counting dimension of the graph. The box-counting dimension is also commonly called the Minkowski dimension.

The basic idea involves covering a set with an $\frac{1}{n} \times \frac{1}{n}$ grid. Then, counting how many boxes from that grid a set hits. As n grows and the grid squares get smaller, more squares will hit the set. The number of squares often grows like a negative power of n, say $N_n \approx n^d$, and d is called the box-counting dimension. For example, the box-counting dimension of a line is 1 and the box-counting dimension of a square is 2. To compute the dimension of the graph of the Weierstrass function, we take the logarithm of this equation and solve for α to get $\alpha = \log(N_n)/\log(n)$ (the base does not matter as long as you use the same base for both logarithms).

(1) Take $b = 2$ and $\alpha = 1/4, 1/2, 3/4$. Plot the graph of $f_{2,\alpha}$ on $[0,1]$ for each of these choices.

(2) Let $N_n(f)$ be the number of boxes from the standard $\frac{1}{n} \times \frac{1}{n}$ grid of squares in the plane that are hit by the graph of f. Show that

$$N_n(f) \approx \sum_{k=1}^{n} n(\max(f, I_k) - \min(f, I_k))$$

where $I_k = [\frac{k-1}{n}, \frac{k}{n}]$. Write a Matlab function that takes f and n as inputs and returns the value of the sum on the right.

(3) The box-counting dimension of the graph of f is defined as

$$\lim_{n\to\infty} \frac{\log N_n(f)}{\log n}.$$

Estimate this limit for the Weierstrass function $f_{2,\alpha}$ for several α values, say $\alpha =, .2, .3, \ldots, .9$. Plot your estimates. Can you formulate a conjecture for what the box-counting dimension is as a function of α? Since $f_{b,\alpha}$ is given by an infinite series, you will have to replace the infinite sum by a finite sum in your experiments. Taking $k = 50$ or 100 should give good results for $n \leq 1{,}000{,}000$.

(4) Repeat your experiments for other values of b, say $b = 3, 4$. For each α, does the box-counting dimension seem the same as before, or does it change when b changes?

Remark: The dimension of the graph of the Weierstrass function is discussed in Chapter 5 of "Fractals in Probability and Analysis"; a PDF of this book is available at:
https://www.math.stonybrook.edu/~waterman/Fall23_MAT331/OtherMaterial/fractalbook_final.pdf