Cantor Julia sets

Vyron Vellis (UTK)
joint works with A. Fletcher (NIU) and D. Stoertz (St. Olaf College)

Special Session on Complex dynamics and Related Areas
AMS Spring Central Sectional Meeting
April 21, 2024

Research partially supported by NSF DMS-1952510 and DMS-2154918

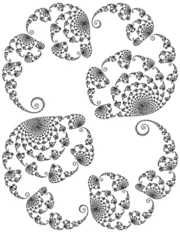
Cantor Julia sets of holomorphic maps

Question: Which Cantor sets are Julia sets of holomorphic maps?

Cantor Julia sets of holomorphic maps

Question: Which Cantor sets are Julia sets of holomorphic maps?

• If c is in the exterior of the Mandelbrot set, then $\mathcal{J}(z^2 + c)$ is a Cantor set. For example,



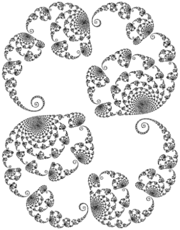
$z^2 + 0.26$

$z^2 + 0.3$

Cantor Julia sets of holomorphic maps

Question: Which Cantor sets are Julia sets of holomorphic maps?

- If c is in the exterior of the Mandelbrot set, then $\mathcal{J}(z^2 + c)$ is a Cantor set. For example,



$z^2 + 0.26$

$z^2 + 0.3$

- The standard Cantor $\mathcal{C}$ set is not the Julia set of a holomorphic map!

Uniformly quasiregular maps

Complex dynamics of holomorphic maps have two important restrictions:

Uniformly quasiregular maps

Complex dynamics of holomorphic maps have two important restrictions:

1. Well-known examples (e.g. standard Cantor set) are absent.

Uniformly quasiregular maps

Complex dynamics of holomorphic maps have two important restrictions:

1. Well-known examples (e.g. standard Cantor set) are absent.
2. **Gehring-Reshetnyak rigidity Theorem:** There are no non-trivial holomorphic maps in dimensions $n \geq 3$.

Uniformly quasiregular maps

Complex dynamics of holomorphic maps have two important restrictions:

1. Well-known examples (e.g. standard Cantor set) are absent.
2. **Gehring-Reshetnyak rigidity Theorem:** There are no non-trivial holomorphic maps in dimensions $n \geq 3$.

- An orientation-preserving map $f : \mathbb{S}^n \to \mathbb{S}^n$ is K-quasiregular if $f \in \mathscr{W}^{1,n}_{\text{loc}}(\mathbb{S}^n)$ and

$$K^{-1} \max_{|h|=1} |f'(x)h| \leq (J_f(x))^{1/n} \leq K \min_{|h|=1} |f'(x)h| \qquad \text{a.e. } x \in \mathbb{S}^n.$$

Uniformly quasiregular maps

Complex dynamics of holomorphic maps have two important restrictions:

1. Well-known examples (e.g. standard Cantor set) are absent.
2. **Gehring-Reshetnyak rigidity Theorem:** There are no non-trivial holomorphic maps in dimensions $n \geq 3$.

- An orientation-preserving map $f : \mathbb{S}^n \to \mathbb{S}^n$ is K-quasiregular if $f \in \mathscr{W}^{1,n}_{\text{loc}}(\mathbb{S}^n)$ and

$$K^{-1} \max_{|h|=1} |f'(x)h| \leq (J_f(x))^{1/n} \leq K \min_{|h|=1} |f'(x)h| \qquad \text{a.e. } x \in \mathbb{S}^n.$$

- An orientation-preserving map $f : \mathbb{S}^n \to \mathbb{S}^n$ is uniformly quasiregular (UQR) if there exists $K \geq 1$ such that f^m is K-quasiregular for all $m \in \mathbb{N}$.

Hyperbolic UQR maps

Hyperbolic UQR maps

- UQR maps are discrete but may not be injective. The branch set

$$\mathcal{B}(f) = \{x \in \mathbb{S}^n : f \text{ is not locally a homeomorphism at } x\}.$$

Example: If $f(z) = z^2$, then $\mathcal{B}(f) = \{0, \infty\}$.

Hyperbolic UQR maps

- UQR maps are discrete but may not be injective. The branch set

$$\mathcal{B}(f) = \{x \in \mathbb{S}^n : f \text{ is not locally a homeomorphism at } x\}.$$

Example: If $f(z) = z^2$, then $\mathcal{B}(f) = \{0, \infty\}$.

- A map $f : \mathbb{S}^n \to \mathbb{S}^n$, $n \geq 2$ is hyperbolic UQR if it is UQR and

$$\mathcal{J}(f) \cap \overline{\bigcup_{m\geq 0} f^m(\mathcal{B}(f))} = \emptyset.$$

Example: The map $f(z) = z^2$ is hyperbolic.

Hyperbolic UQR maps

- UQR maps are discrete but may not be injective. The branch set

$$\mathcal{B}(f) = \{x \in \mathbb{S}^n : f \text{ is not locally a homeomorphism at } x\}.$$

Example: If $f(z) = z^2$, then $\mathcal{B}(f) = \{0, \infty\}$.

- A map $f : \mathbb{S}^n \to \mathbb{S}^n$, $n \geq 2$ is hyperbolic UQR if it is UQR and

$$\mathcal{J}(f) \cap \overline{\bigcup_{m \geq 0} f^m(\mathcal{B}(f))} = \emptyset.$$

Example: The map $f(z) = z^2$ is hyperbolic.

Question

Which Cantor sets are Julia sets for (hyperbolic) UQR maps?

(Quasi)-self-similarity

(Quasi)-self-similarity

Definition

A Cantor set $X \subset \mathbb{S}^n$ is self-similar if $X = X_1 \cup \cdots \cup X_k$ where X_i are mutually disjoint rescaled copies of X.

(Quasi)-self-similarity

Definition

A Cantor set $X \subset \mathbb{S}^n$ is self-similar if $X = X_1 \cup \cdots \cup X_k$ where X_i are mutually disjoint rescaled copies of X.

Definition

A Cantor set $X \subset \mathbb{S}^n$ is quasi-self-similar if for any $x \in X$ and $r \in (0, \operatorname{diam} X)$ there exists $x \in E_{x,r} \subset X$ such that

$$\operatorname{dist}(E_{x,r}, X \setminus E_{x,r}) \simeq r \simeq \operatorname{diam} E_{x,r}.$$

(Quasi)-self-similarity

Definition

A Cantor set $X \subset \mathbb{S}^n$ is self-similar if $X = X_1 \cup \cdots \cup X_k$ where X_i are mutually disjoint rescaled copies of X.

Definition

A Cantor set $X \subset \mathbb{S}^n$ is quasi-self-similar if for any $x \in X$ and $r \in (0, \operatorname{diam} X)$ there exists $x \in E_{x,r} \subset X$ such that

$$\operatorname{dist}(E_{x,r}, X \setminus E_{x,r}) \simeq r \simeq \operatorname{diam} E_{x,r}.$$

- **QS uniformization of Cantor sets (David-Semmes 1998)**

A Cantor set X is quasi-self-similar if and only if $X \approx^{QS} \mathcal{C}$.

Examples

Examples

- The Cantor set $C[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots]$ is self-similar.

Examples

- The Cantor set $C[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots]$ is self-similar.

- More generally, if (a_n) is a periodic sequence in $(0, 1)$, then $C[a_1, a_2, a_3, \dots]$ is self-similar.

Examples

- The Cantor set $C[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots]$ is self-similar.
- More generally, if (a_n) is a periodic sequence in $(0, 1)$, then $C[a_1, a_2, a_3, \dots]$ is self-similar.
- If (a_n) is a sequence in $(0, 1)$ that does not approach 0 or 1, then $C[a_1, a_2, a_3, \dots]$ is quasi-self-similar.

Examples

- The Cantor set $C[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots]$ is self-similar.

- More generally, if (a_n) is a periodic sequence in $(0,1)$, then $C[a_1, a_2, a_3, \dots]$ is self-similar.

- If (a_n) is a sequence in $(0,1)$ that does not approach 0 or 1, then $C[a_1, a_2, a_3, \dots]$ is quasi-self-similar.

- The Cantor set $C[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots] \sqcup C[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots]$ is quasi-self-similar but not self-similar.

Examples

- The Cantor set $C[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots]$ is self-similar.

- More generally, if (a_n) is a periodic sequence in $(0, 1)$, then $C[a_1, a_2, a_3, \ldots]$ is self-similar.

- If (a_n) is a sequence in $(0, 1)$ that does not approach 0 or 1, then $C[a_1, a_2, a_3, \ldots]$ is quasi-self-similar.

- The Cantor set $C[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots] \sqcup C[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots]$ is quasi-self-similar but not self-similar.

- The Cantor set $C[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots]$ is not quasi-self-similar.

Geometric obstructions for Cantor Julia sets

Geometric obstructions for Cantor Julia sets

- **Fletcher-Nicks (2011)** The Julia set of a UQR map is uniformly perfect.

Example: $C[\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots]$ can not be the Julia set of a UQR map.

Geometric obstructions for Cantor Julia sets

- **Fletcher-Nicks (2011)** The Julia set of a UQR map is uniformly perfect.

Example: $C[\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots]$ can not be the Julia set of a UQR map.

- **Fletcher-V (2021)** If the Julia set of a hyperbolic UQR map is a Cantor set, then it is quasi-self-similar.

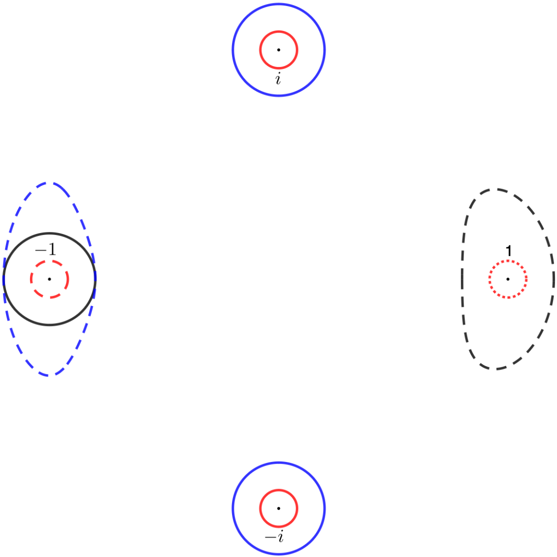
Example: $C[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots]$ can not be the Julia set of a hyperbolic UQR map.

Examples of planar (quasi-)self-similar Julia sets

Examples of planar (quasi-)self-similar Julia sets

- **Iwaniec-Martin (1996)** The standard Cantor set $\mathcal{C}$ is the Julia set of a hyperbolic UQR map of $\mathbb{S}^2$.

Conformal trap method



Examples of planar (quasi-)self-similar Julia sets

- **Iwaniec-Martin (1996)** The standard Cantor set $\mathcal{C}$ is the Julia set of a hyperbolic UQR map of $\mathbb{S}^2$.

- **Fletcher (2019)** If $X \subset \mathbb{S}^2$ is a self-similar Cantor set, then it is a Julia set of a hyperbolic UQR map of $\mathbb{S}^2$.

Examples of planar (quasi-)self-similar Julia sets

- **Iwaniec-Martin (1996)** The standard Cantor set $\mathcal{C}$ is the Julia set of a hyperbolic UQR map of $\mathbb{S}^2$.

- **Fletcher (2019)** If $X \subset \mathbb{S}^2$ is a self-similar Cantor set, then it is a Julia set of a hyperbolic UQR map of $\mathbb{S}^2$.

Theorem (Fletcher-V 2021)

A Cantor set $X \subset \mathbb{S}^2$ is the Julia set of a hyperbolic UQR map of $\mathbb{S}^2$ if and only if it is quasi-self-similar.

Examples of planar (quasi-)self-similar Julia sets

- **Iwaniec-Martin (1996)** The standard Cantor set $\mathcal{C}$ is the Julia set of a hyperbolic UQR map of $\mathbb{S}^2$.

- **Fletcher (2019)** If $X \subset \mathbb{S}^2$ is a self-similar Cantor set, then it is a Julia set of a hyperbolic UQR map of $\mathbb{S}^2$.

Theorem (Fletcher-V 2021)

A Cantor set $X \subset \mathbb{S}^2$ is the Julia set of a hyperbolic UQR map of $\mathbb{S}^2$ if and only if it is quasi-self-similar.

- **QC uniformization of Cantor sets in $\mathbb{S}^2$ (MacManus 1999)**

If $X \subset \mathbb{S}^2$ is a quasi-self-similar Cantor set, then there exists a quasiconformal $f : \mathbb{S}^2 \to \mathbb{S}^2$ such that $f(X) = \mathcal{C}$.

Tame Cantor sets

Tame Cantor sets

- A Cantor set $X \subset \mathbb{S}^n$ is tame if there exists a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $f(X) = \mathcal{C}$.

Tame Cantor sets

- A Cantor set $X \subset \mathbb{S}^n$ is tame if there exists a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $f(X) = \mathcal{C}$.

- **Moise (1918)** Every Cantor set in dimensions 1,2 is tame.

Tame Cantor sets

- A Cantor set $X \subset \mathbb{S}^n$ is tame if there exists a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $f(X) = \mathcal{C}$.

- **Moise (1918)** Every Cantor set in dimensions 1,2 is tame.

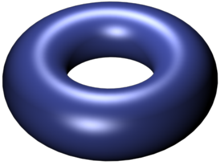
- **Fletcher-V (2021)** If $X \subset \mathbb{S}^3$ is a self-similar and tame* Cantor set, then it is a Julia set of a hyperbolic UQR map of $\mathbb{S}^3$.

Wild Cantor sets

- A Cantor set $X \subset \mathbb{S}^n$ is wild if it is not tame.

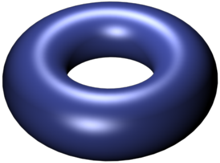
Wild Cantor sets

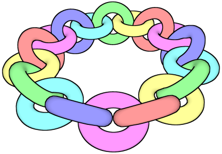
- A Cantor set $X \subset \mathbb{S}^n$ is wild if it is not tame.



Wild Cantor sets

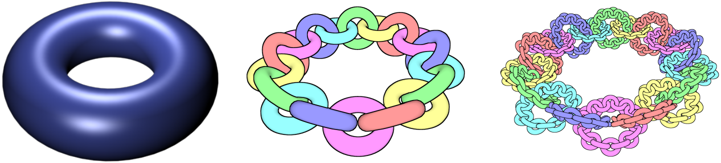
- A Cantor set $X \subset \mathbb{S}^n$ is wild if it is not tame.





Wild Cantor sets

- A Cantor set $X \subset \mathbb{S}^n$ is wild if it is not tame.



Antoine necklace (Image from Wikipedia)

- The Antoine necklace is a self-similar wild Cantor set.

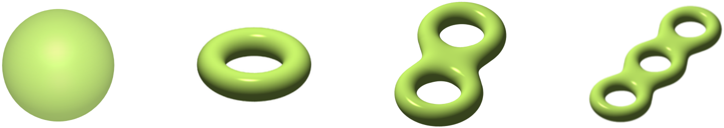
Genus of a Cantor set in $\mathbb{S}^3$

Let $X \subset \mathbb{S}^3$ be a Cantor set.

Genus of a Cantor set in $\mathbb{S}^3$

Let $X \subset \mathbb{S}^3$ be a Cantor set.

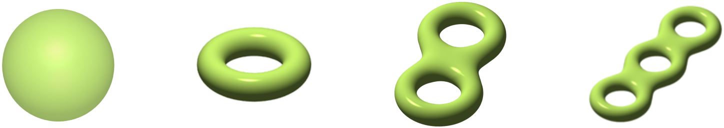
- **Armentrout (1966)** There exists a decreasing sequence $(M_n)_{n\in\mathbb{N}}$ of handlebodies such that $X = \bigcap_{n=1}^{\infty} M_n$.



Genus of a Cantor set in $\mathbb{S}^3$

Let $X \subset \mathbb{S}^3$ be a Cantor set.

- **Armentrout (1966)** There exists a decreasing sequence $(M_n)_{n\in\mathbb{N}}$ of handlebodies such that $X = \bigcap_{n=1}^{\infty} M_n$.

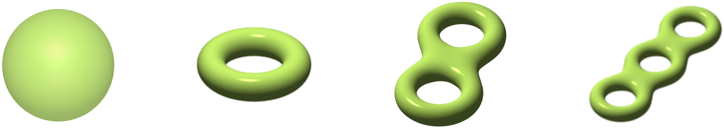


▶ Defining sequences are not unique! Denote by $\mathcal{D}(X)$ all such defining sequences for X.

Genus of a Cantor set in $\mathbb{S}^3$

Let $X \subset \mathbb{S}^3$ be a Cantor set.

- **Armentrout (1966)** There exists a decreasing sequence $(M_n)_{n\in\mathbb{N}}$ of handlebodies such that $X = \bigcap_{n=1}^{\infty} M_n$.



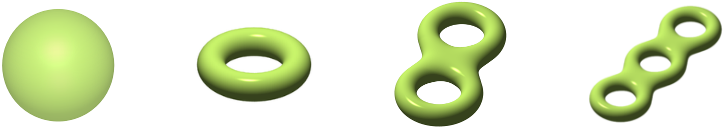
- Defining sequences are not unique! Denote by $\mathcal{D}(X)$ all such defining sequences for X.
- **Željko (2005):** The genus of X is

$$g(X) = \inf_{(M_n)\in\mathcal{D}(X)} \sup_{n\in\mathbb{N}} g(M_n).$$

Genus of a Cantor set in $\mathbb{S}^3$

Let $X \subset \mathbb{S}^3$ be a Cantor set.

- **Armentrout (1966)** There exists a decreasing sequence $(M_n)_{n\in\mathbb{N}}$ of handlebodies such that $X = \bigcap_{n=1}^{\infty} M_n$.



- Defining sequences are not unique! Denote by $\mathcal{D}(X)$ all such defining sequences for X.
- **Željko (2005):** The genus of X is

$$g(X) = \inf_{(M_n)\in\mathcal{D}(X)} \sup_{n\in\mathbb{N}} g(M_n).$$

- Given $x \in X$ we can define $g_x(X)$ the local genus of X at x.

Examples of Cantor sets of any genus

Examples of Cantor sets of any genus

- A Cantor set $X \subset \mathbb{S}^3$ is tame (e.g. $\mathcal{C}$) if and only if $g(X) = 0$.

Examples of Cantor sets of any genus

- A Cantor set $X \subset \mathbb{S}^3$ is tame (e.g. $\mathcal{C}$) if and only if $g(X) = 0$.

- A Cantor set $X \subset \mathbb{S}^3$ is wild (e.g. the Antoine necklace) if and only if $g(X) \geq 1$.

Examples of Cantor sets of any genus

- A Cantor set $X \subset \mathbb{S}^3$ is tame (e.g. $\mathcal{C}$) if and only if $g(X) = 0$.

- A Cantor set $X \subset \mathbb{S}^3$ is wild (e.g. the Antoine necklace) if and only if $g(X) \geq 1$.

- If X is the Antoine necklace, then $g(X) = 1$. Also $g_x(X) = 1$ for all $x \in X$.

Examples of Cantor sets of any genus

- A Cantor set $X \subset \mathbb{S}^3$ is tame (e.g. $\mathcal{C}$) if and only if $g(X) = 0$.

- A Cantor set $X \subset \mathbb{S}^3$ is wild (e.g. the Antoine necklace) if and only if $g(X) \geq 1$.

- If X is the Antoine necklace, then $g(X) = 1$. Also $g_x(X) = 1$ for all $x \in X$.

- **Željko (2005)** For each $n \in \mathbb{N}$ there exists a Cantor set $X_n \subset \mathbb{S}^3$ that has $g(X_n) = n$.

Examples of Cantor sets of any genus

- A Cantor set $X \subset \mathbb{S}^3$ is tame (e.g. $\mathcal{C}$) if and only if $g(X) = 0$.
- A Cantor set $X \subset \mathbb{S}^3$ is wild (e.g. the Antoine necklace) if and only if $g(X) \geq 1$.
- If X is the Antoine necklace, then $g(X) = 1$. Also $g_x(X) = 1$ for all $x \in X$.
- **Željko (2005)** For each $n \in \mathbb{N}$ there exists a Cantor set $X_n \subset \mathbb{S}^3$ that has $g(X_n) = n$.
- However, in Željko's example, there is only one point $x \in X_n$ with $g_x(X_n) = n$.

Topological obstructions for Cantor Julia sets

Topological obstructions for Cantor Julia sets

Theorem (Fletcher-Stoertz-V arXiv)

Suppose that X is a Cantor set and the Julia set of a hyperbolic UQR map of $\mathbb{S}^3$.

Topological obstructions for Cantor Julia sets

Theorem (Fletcher-Stoertz-V arXiv)

Suppose that X is a Cantor set and the Julia set of a hyperbolic UQR map of $\mathbb{S}^3$.

1. *The genus $g(X) < \infty$.*

Topological obstructions for Cantor Julia sets

Theorem (Fletcher-Stoertz-V arXiv)

Suppose that X is a Cantor set and the Julia set of a hyperbolic UQR map of $\mathbb{S}^3$.

1. *The genus $g(X) < \infty$.*
2. *The set*
$$\{x \in X : g_x(X) = g(X)\}$$
is dense in X.

Topological obstructions for Cantor Julia sets

Theorem (Fletcher-Stoertz-V arXiv)

Suppose that X is a Cantor set and the Julia set of a hyperbolic UQR map of $\mathbb{S}^3$.

1. *The genus $g(X) < \infty$.*
2. *The set*
$$\{x \in X : g_x(X) = g(X)\}$$
is dense in X.

Example: If $\mathcal{A}$ is the Antoine necklace, then $\mathcal{C} \sqcup \mathcal{A}$ is quasi-self-similar but is not the Julia set of a hyperbolic UQR map.

Topological obstructions for Cantor Julia sets

Theorem (Fletcher-Stoertz-V arXiv)

Suppose that X is a Cantor set and the Julia set of a hyperbolic UQR map of $\mathbb{S}^3$.

1. *The genus $g(X) < \infty$.*
2. *The set*
$$\{x \in X : g_x(X) = g(X)\}$$
is dense in X.

Example: If $\mathcal{A}$ is the Antoine necklace, then $\mathcal{C} \sqcup \mathcal{A}$ is quasi-self-similar but is not the Julia set of a hyperbolic UQR map.

Example: Željko's example is quasi-self-similar but not the Julia set of a hyperbolic UQR map.

Examples of Julia Cantor sets of any genus

Examples of Julia Cantor sets of any genus

- **Iwaniec-Martin (1996)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 0$ for all $x \in \mathcal{J}(f)$. (Here $\mathcal{J}(f) = \mathcal{C}$)

Examples of Julia Cantor sets of any genus

- **Iwaniec-Martin (1996)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 0$ for all $x \in \mathcal{J}(f)$. (Here $\mathcal{J}(f) = \mathcal{C}$)

- **Fletcher-Wu (2015)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 1$ for all $x \in \mathcal{J}(f)$. (Here $\mathcal{J}(f)$ is the Antoine necklace)

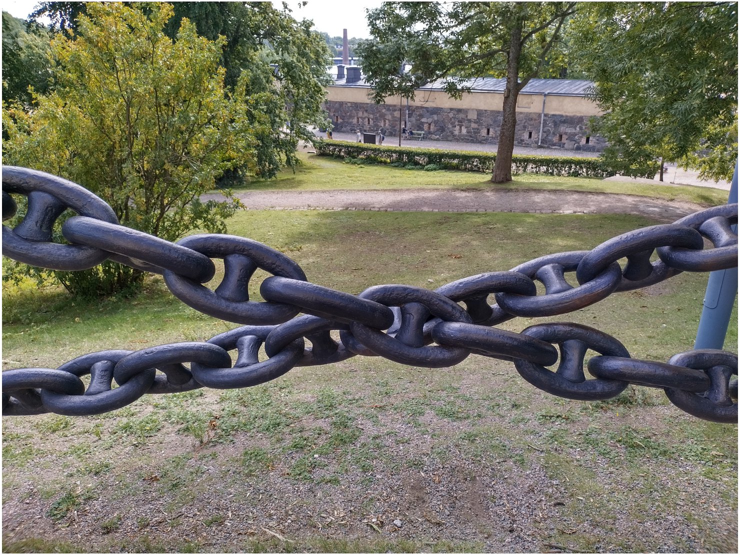
Examples of Julia Cantor sets of any genus

- **Iwaniec-Martin (1996)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 0$ for all $x \in \mathcal{J}(f)$. (Here $\mathcal{J}(f) = \mathcal{C}$)

- **Fletcher-Wu (2015)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 1$ for all $x \in \mathcal{J}(f)$. (Here $\mathcal{J}(f)$ is the Antoine necklace)

- **Fletcher-Stoertz (2023)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 2$ for all $x \in \mathcal{J}(f)$.

Examples of Julia Cantor sets of any genus



Examples of Julia Cantor sets of any genus

- **Iwaniec-Martin (1996)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 0$ for all $x \in \mathcal{J}(f)$. (Here $\mathcal{J}(f) = \mathcal{C}$)

- **Fletcher-Wu (2015)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 1$ for all $x \in \mathcal{J}(f)$. (Here $\mathcal{J}(f)$ is the Antoine necklace)

- **Fletcher-Stoertz (2023)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 2$ for all $x \in \mathcal{J}(f)$.

Examples of Julia Cantor sets of any genus

- **Iwaniec-Martin (1996)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 0$ for all $x \in \mathcal{J}(f)$. (Here $\mathcal{J}(f) = \mathcal{C}$)

- **Fletcher-Wu (2015)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 1$ for all $x \in \mathcal{J}(f)$. (Here $\mathcal{J}(f)$ is the Antoine necklace)

- **Fletcher-Stoertz (2023)** There exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = 2$ for all $x \in \mathcal{J}(f)$.

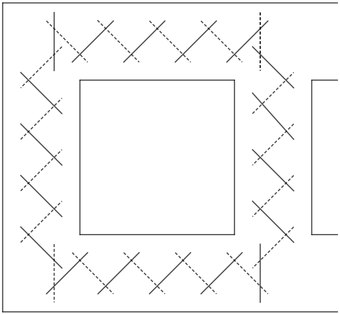
Theorem (Fletcher-Stoertz-V arXiv)

For each $n \in \mathbb{N}$ there exists a hyperbolic UQR map f of $\mathbb{S}^3$ such that $g_x(\mathcal{J}(f)) = n$ for all $x \in \mathcal{J}(f)$.

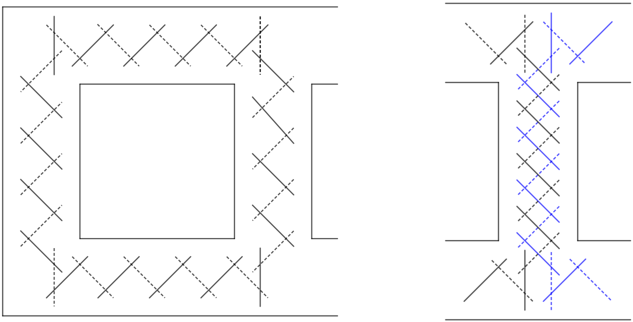
Construction for $n = 3$: 1st step

Ladder: Let T be the genus 3 thickening of the set bellow.

Construction for $n = 3$: 2nd step



Construction for $n = 3$: 2nd step



Construction of the UQR map

- There exists a quasiregular map

$$F : T \setminus \bigcup_i \mathrm{int}(T_i) \to B \setminus \mathrm{int}(T)$$

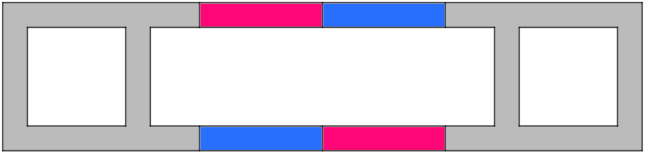
so that $F|\partial T_i$ is a similarity for all i.

Construction of the UQR map

- There exists a quasiregular map

$$F : T \setminus \bigcup_i \mathrm{int}(T_i) \to B \setminus \mathrm{int}(T)$$

so that $F|\partial T_i$ is a similarity for all i.

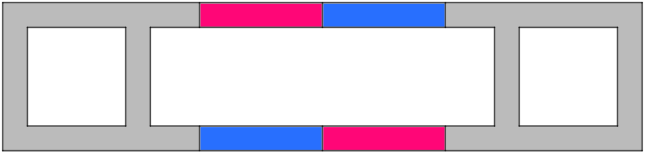


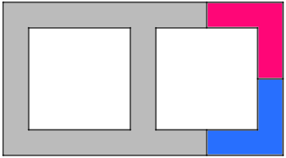
Construction of the UQR map

- There exists a quasiregular map

$$F : T \setminus \bigcup_i \operatorname{int}(T_i) \to B \setminus \operatorname{int}(T)$$

so that $F|\partial T_i$ is a similarity for all i.



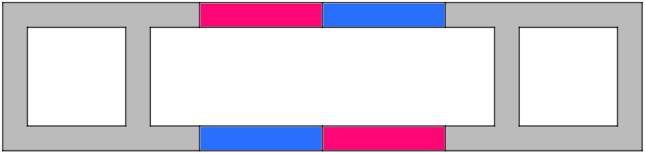


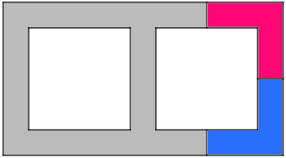
Construction of the UQR map

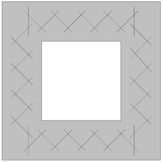
- There exists a quasiregular map

$$F : T \setminus \bigcup_i \mathrm{int}(T_i) \to B \setminus \mathrm{int}(T)$$

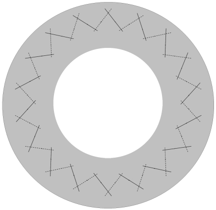
so that $F|\partial T_i$ is a similarity for all i.



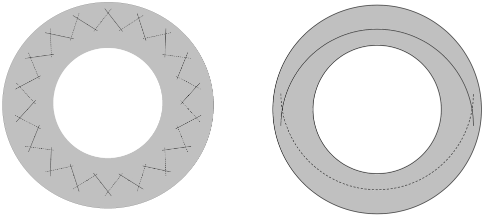




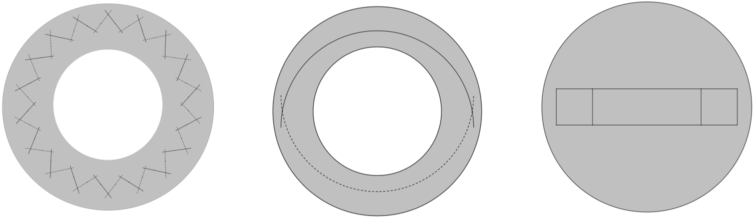
Construction of the UQR map (cont.)



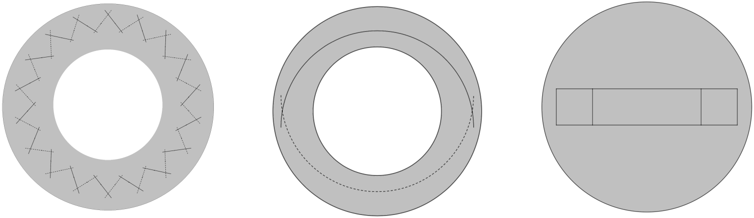
Construction of the UQR map (cont.)



Construction of the UQR map (cont.)



Construction of the UQR map (cont.)



$$\begin{array}{ccccccccc} \mathbb{R}^3 = & \mathbb{R}^3 \setminus B & \cup & B \setminus T & \cup & T \setminus \bigcup_i T_i & \cup & \bigcup_i T_i \\ & \downarrow \text{UQR map} & & \downarrow \text{BE extension} & & \downarrow F & & \downarrow \text{similarities} \\ \mathbb{R}^3 = & \mathbb{R}^3 \setminus B' & \cup & B' \setminus B & \cup & B \setminus T & \cup & T \end{array}$$

Julia Cantor sets of non-constant local genus

Julia Cantor sets of non-constant local genus

Theorem (Fletcher-Stoertz-V arXiv)

For each $n \in \mathbb{N}$ there exists a hyperbolic UQR map of $\mathbb{S}^3$ such that

$$\{g_x(\mathcal{J}(f)) : x \in \mathcal{J}(f)\} = \{0, n\}.$$

Julia Cantor sets of non-constant local genus

Theorem (Fletcher-Stoertz-V arXiv)

For each $n \in \mathbb{N}$ there exists a hyperbolic UQR map of $\mathbb{S}^3$ such that

$$\{g_x(\mathcal{J}(f)) : x \in \mathcal{J}(f)\} = \{0, n\}.$$

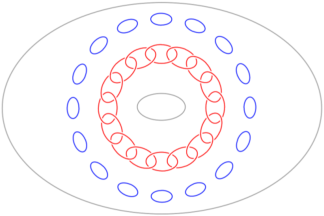


Figure: The case $n = 1$.

Cantor Julia sets in $\mathbb{S}^n$ with $n \geq 4$

Cantor Julia sets in $\mathbb{S}^n$ with $n \geq 4$

- Wild Cantor sets exist in $\mathbb{S}^n$ for all $n \geq 3$. However the following are unknown:

Cantor Julia sets in $\mathbb{S}^n$ with $n \geq 4$

- Wild Cantor sets exist in $\mathbb{S}^n$ for all $n \geq 3$. However the following are unknown:
  1. Are there self-similar wild Cantor sets in $\mathbb{S}^n$ when $n \geq 4$?
  2. Are there quasi-self-similar wild Cantor sets in $\mathbb{S}^n$ when $n \geq 5$?

Cantor Julia sets in $\mathbb{S}^n$ with $n \geq 4$

▶ Wild Cantor sets exist in $\mathbb{S}^n$ for all $n \geq 3$. However the following are unknown:

1. Are there self-similar wild Cantor sets in $\mathbb{S}^n$ when $n \geq 4$?
2. Are there quasi-self-similar wild Cantor sets in $\mathbb{S}^n$ when $n \geq 5$?

• **Pankka-Wu (arxiv)** There exists a quasi-self-similar Cantor set in $\mathbb{S}^4$ and it is a Julia set of a (hyperbolic?) UQR map.

Cantor Julia sets in $\mathbb{S}^n$ with $n \geq 4$

- Wild Cantor sets exist in $\mathbb{S}^n$ for all $n \geq 3$. However the following are unknown:
  1. Are there self-similar wild Cantor sets in $\mathbb{S}^n$ when $n \geq 4$?
  2. Are there quasi-self-similar wild Cantor sets in $\mathbb{S}^n$ when $n \geq 5$?

• **Pankka-Wu (arxiv)** There exists a quasi-self-similar Cantor set in $\mathbb{S}^4$ and it is a Julia set of a (hyperbolic?) UQR map.

Theorem (Fletcher-V 2021)

If $X \subset \mathbb{S}^n$ is a quasi-self-similar Cantor set, then it is the Julia set of some hyperbolic UQR map of $\mathbb{S}^{n+1}$.

Cantor Julia sets in $\mathbb{S}^n$ with $n \geq 4$

- Wild Cantor sets exist in $\mathbb{S}^n$ for all $n \geq 3$. However the following are unknown:
  1. Are there self-similar wild Cantor sets in $\mathbb{S}^n$ when $n \geq 4$?
  2. Are there quasi-self-similar wild Cantor sets in $\mathbb{S}^n$ when $n \geq 5$?

- **Pankka-Wu (arxiv)** There exists a quasi-self-similar Cantor set in $\mathbb{S}^4$ and it is a Julia set of a (hyperbolic?) UQR map.

Theorem (Fletcher-V 2021)

If $X \subset \mathbb{S}^n$ is a quasi-self-similar Cantor set, then it is the Julia set of some hyperbolic UQR map of $\mathbb{S}^{n+1}$.

- **QC uniformization of Cantor sets in $\mathbb{S}^n$ (V 2021)**

If $X \subset \mathbb{S}^n$ is a quasi-self-similar Cantor set, then there exists a quasiconformal $f : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ such that $f(X) = \mathcal{C}$.

Open questions

Open questions

Question

Let $X \subset \mathbb{S}^n$ be self-similar. Is there a hyperbolic UQR $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $\mathcal{J}(f) = X$?

Open questions

Question

Let $X \subset \mathbb{S}^n$ be self-similar. Is there a hyperbolic UQR $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $\mathcal{J}(f) = X$?

Question

Let $X \subset \mathbb{S}^n$ be quasi-self-similar. Is there a UQR $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $\mathcal{J}(f) = X$?

Open questions

Question

Let $X \subset \mathbb{S}^n$ be self-similar. Is there a hyperbolic UQR $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $\mathcal{J}(f) = X$?

Question

Let $X \subset \mathbb{S}^n$ be quasi-self-similar. Is there a UQR $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $\mathcal{J}(f) = X$?

Thank you for your attention.