# Mandelbrot Breadcrumbs

## Engaging Undergraduates in Complex Dynamics Research

Daniel Stoertz
*they/them/their*

St. Olaf College

AMS Spring Central Sectional Meeting, April 21, 2024

1. Overview of the Project
2. Good for Undergraduates
   - Starting from Analysis Basics
   - Developing and Utilizing Coding Skills
   - Visual Intuition and Beauty
3. A Comprehensive Anecdote

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- Team of two undergraduate researchers, Murali Meyer and Mike Wang.

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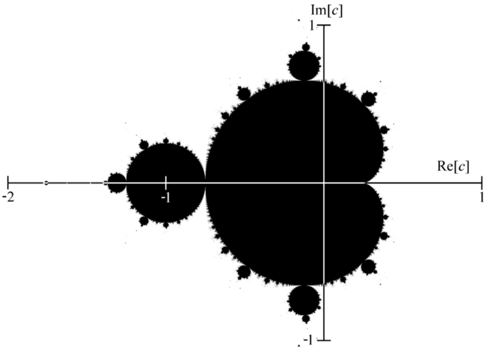

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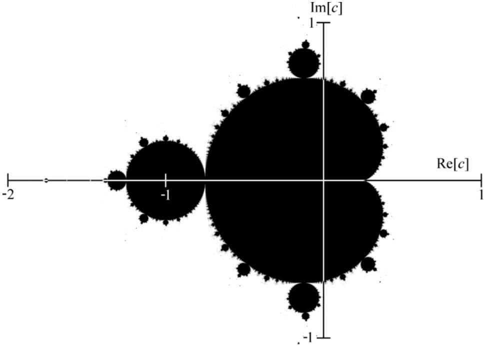
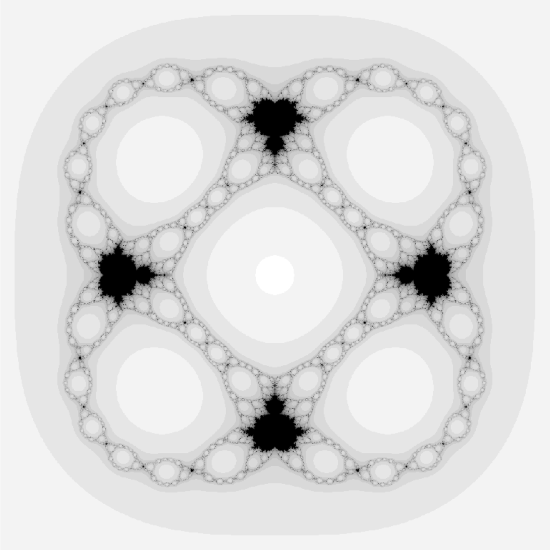



- This idea can be applied to other parameter families.

# Background

McMullen Map, n=5



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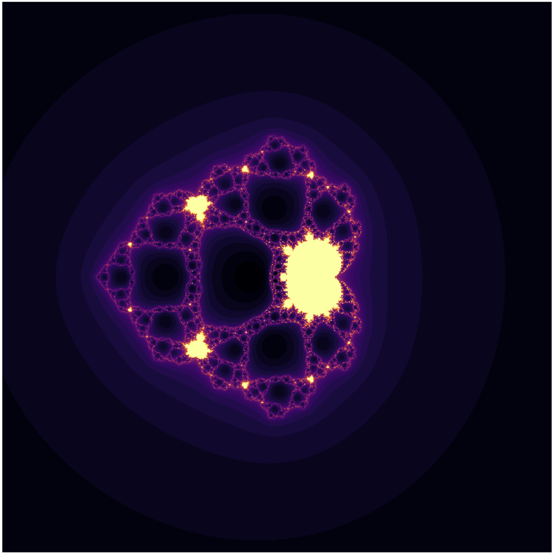
- In all cases, baby $\mathcal{M}$ are located by tracking behavior of segments of certain annuli under one application of $f$.

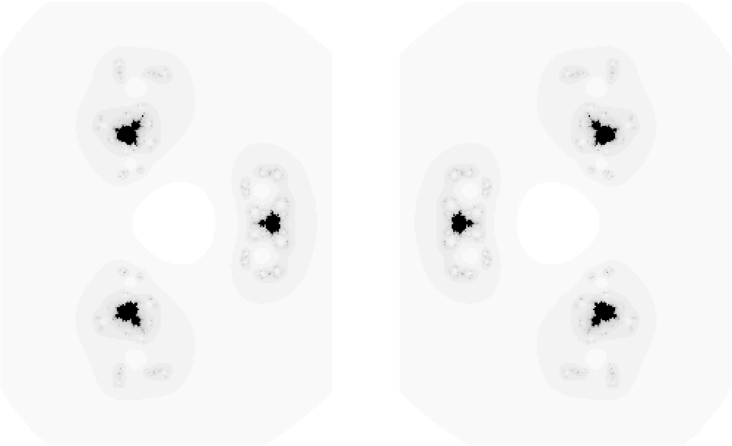
Jang-So set, z^n+a/z^d, n=2, d=3



# Boyd-Mitchell set, z^n+a/z^n+c, n=3, a=0.5, c=variable

Boundedness Locus of V+

Boundedness Locus of V-



# Results

**Theorem (Meyer, S., Wang)**

*Let $f_{a,c}(z) = z^n + \frac{a}{z^d} + c$, with $n, d \geq 3$. Fix real $a \geq k(n,d)$. Then there exist $n + d$ baby $\mathcal{M}$ in the boundedness locus in the c-parameter plane.*

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# Starting from Analysis Basics

Some sample problems from the first week:

2. Express the following complex numbers in the form $a + bi$.
   (a) $(2 + 3i) + (4 + i)$
   (b) $(2 + 3i)(4 + i)$
   (c) $\dfrac{2 + 3i}{4 + i}$
   (d) $\dfrac{1}{4 + i} - \dfrac{1}{2 + 3i}$

9. Use the epsilon-delta definition to prove the following limits.
   (a) $\lim_{z \to 2+i} 2z - 3 = 1 + 2i$
   (b) $\lim_{z \to i} z^2 = -1$
   (c) $\lim_{z \to 2i} \dfrac{1}{z} = -\dfrac{1}{2} i$

6. Find a conformal mapping that maps each of the following regions onto the unit disk $B_0(1)$.
   (a) The upper-half unit disk $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0, |z| < 1\}$. *Note: $z^2$ doesn't work because it says $\operatorname{Im}(z) > 0$, and this would miss the non-negative real axis segment.*
   (b) The sector $\{z \in \mathbb{C} : |\arg(z) < \pi/4\}$.
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Sample of a proof they wrote towards the end of the project:

*Proof.* Recall that

$$\Gamma = \left\{ z^4 + \frac{a}{z^3} + c \mid z = 1.8a^{\frac{4}{7}} e^{i\theta},\ -\pi < \theta \leq \pi \right\}.$$

Let $z \in \Gamma$. We then have that

$$\begin{aligned}
|z| &= \left| (ka^{\frac{n}{m}} e^{i\theta})^n + \frac{a}{(ka^{\frac{n}{m}} e^{i\theta})^d} + c \right| \\
&\geq \left| |(ka^{\frac{n}{m}} e^{i\theta})^n| - \left| \frac{a}{(ka^{\frac{n}{m}} e^{i\theta})^d} \right| - |c| \right| \\
&= \left| k^n a^{\frac{n^2}{m}} - \frac{1}{k^d a^{\frac{nd-m}{m}}} - |c| \right| \\
&= \left| k^n a^{\frac{n^2}{m}} - k^{-d} a^{\frac{m-nd}{m}} - |c| \right| \\
&> \left| k^n a^{\frac{n^2}{m}} - k^{-d} a^{\frac{m-nd}{m}} - \sqrt{8} a^{\frac{n}{m}} \right|
\end{aligned}$$

We wish to show that

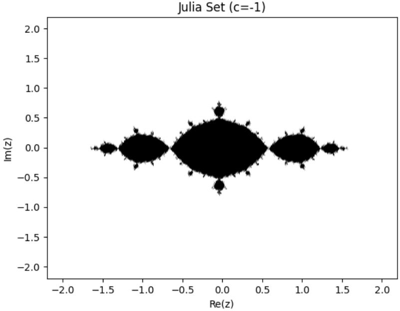
$$\left| k^n a^{\frac{n^2}{m}} - k^{-d} a^{\frac{m-nd}{m}} - \sqrt{8} a^{\frac{n}{m}} \right| > ka^{\frac{n}{m}}.$$

Since $ka^{\frac{n}{m}}$ is positive, it suffices to show

$$k^n a^{\frac{n^2}{m}} - k^{-d} a^{\frac{m-nd}{m}} - \sqrt{8} a^{\frac{n}{m}} > ka^{\frac{n}{m}}.$$

# Developing and Utilizing Coding Skills

Early image using code written from scratch:



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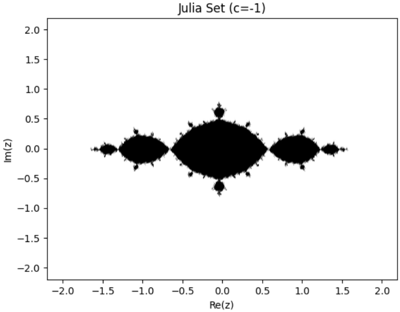
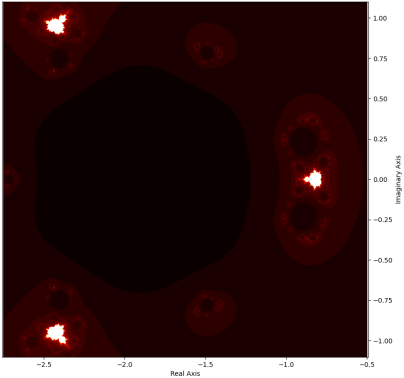



Image generated by code we adapted from online sources:



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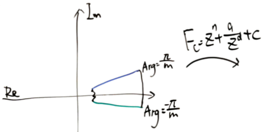


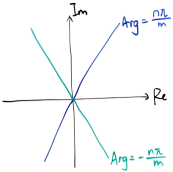


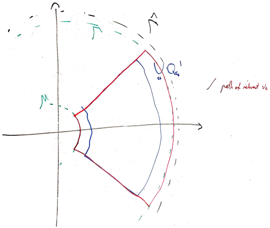

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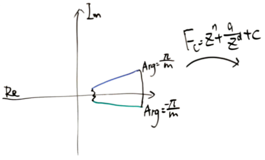


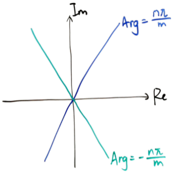


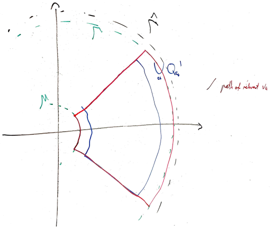

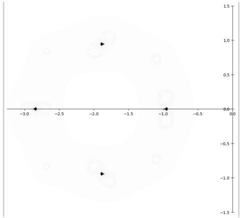

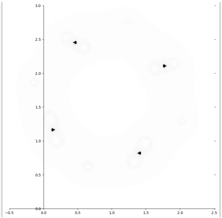

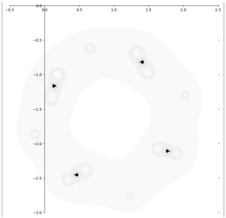

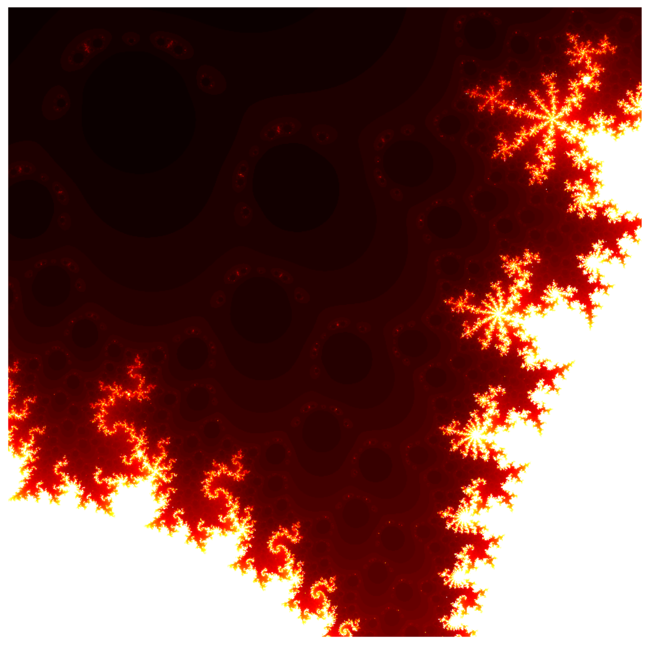

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## Generalized baby Mandelbrot sets adorned with halos in families of rational maps

HyeGyong Jang[a], YongNam So[a] and Sebastian M. Marotta[b]

[a]Faculty of Mathematics, University of Science Pyongyang, Pyongyang, Korea; [b]Department of Mathematics, Boston University, Boston, MA, USA

**ABSTRACT**

We consider the family of rational maps given by $F_\lambda(z) = z^n + \lambda/z^d$ where $n, d \in \mathbb{N}$ with $1/n + 1/d < 1$, the variable $z \in \widehat{\mathbb{C}}$ and the parameter $\lambda \in \mathbb{C}$. It is known [1] that when $n = d \geq 3$ there are $n - 1$ small copies of the Mandelbrot set symmetrically located around the origin in the parameter $\lambda$−plane. These baby Mandelbrot sets have 'antennas' attached to the boundaries of Sierpiński holes. Sierpiński holes are open simply connected subsets of the parameter space for which the Julia sets of $F_\lambda$ are Sierpiński curves. In this paper we generalize the symmetry properties of $F_\lambda$ and the existence of the $n - 1$ baby Mandelbrot sets to the case when $1/n + 1/d < 1$ where $n$ is not necessarily equal to $d$.

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Julia sets; Mandelbrot set; rational maps; complex dynamics

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37f10

To apply the machinery of Douady and Hubbard, for each parameter value $\lambda$ there must exist a neighborhood $U'_\lambda$ of the critical number such that the family $f_\lambda : U'_\lambda \to f(U'_\lambda) = U_\lambda$ is polynomial-like of degree 2.

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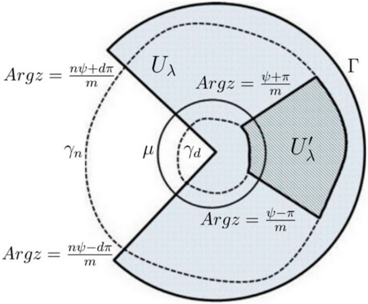
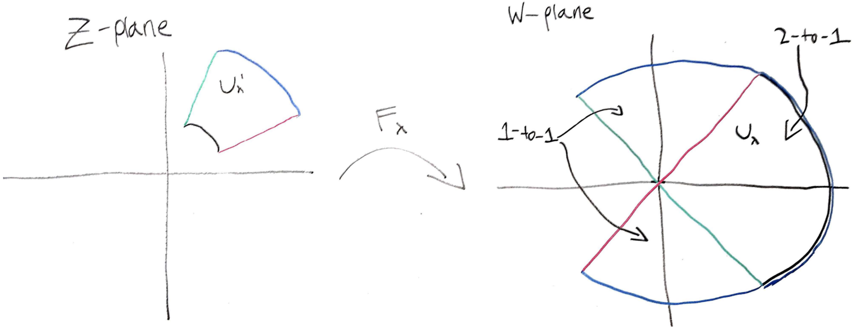
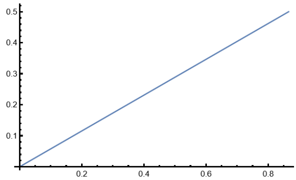



Z-plane
$U_x'$
$F_x$
W-plane
2-to-1
1-to-1
$U_x$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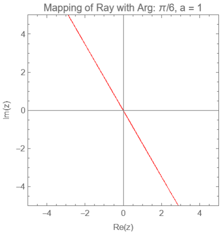
Mapping of Ray with Arg: π/6, a = 1
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−2
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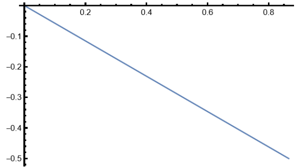
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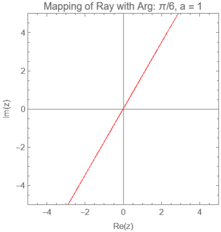
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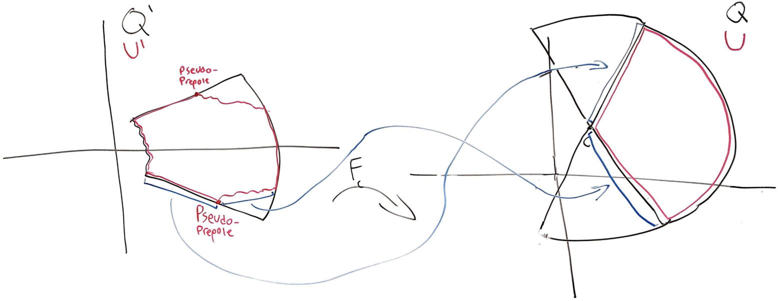
Q'
U'
Pseudo-
Prepole
Pseudo-
Prepole
F_c
Q
U

# References


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