Phragmén-Lindelöf Principles and Julia Limiting Directions of Quasiregular Maps

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Outline

Julia Limiting Directions

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a transcendental entire function.

- The iterates of f are defined by $f^1 = f$, $f^k = f \circ f^{k-1}$
- The Julia set $J(f)$ is the set of chaotic behavior and can be defined via a blowing up property
- From the Great Picard Theorem, $J(f)$ is unbounded.

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- The Julia set $J(f)$ is the set of chaotic behavior and can be defined via a blowing up property
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Definition

Let f be a transcendental entire function. The set of Julia limiting directions is $L(f) = \{e^{i\theta} : \lim_{m\to\infty} \arg(z_m) = \theta$ for a sequence $(z_m)_{m=1}^{\infty} \subset J(f)$ with $|z_m|$ increasing and $\lim_{m\to\infty} z_m = \infty\} \subset S^1$.

$f(z) = e^z$

Example

Consider $f(z) = e^z$. Then, $J(f) = \mathbb{C}$ and $L(f) = S^1$.

Figure: $J(e^z)$

$f(z) = e^z/5$

Example

Consider $f(z) = e^z/5$. Then, $J(f)$ is a Cantor bouquet and $L(f) = \{z \in S^1 : Re(z) \geq 1\}$.

Figure: $J(e^z/5)$

Measure of $L(f)$

Theorem (Qiao, 2001)

Let f be a transcendental entire function of lower order $\lambda < \infty$. Then there exists a closed interval $I \subset S^1$ such that $I \subset L(f)$ and $m(I) \geq \pi / \max(1/2, \lambda)$. In particular, if $\lambda < 1/2$, then $L(f) = S^1$.

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Theorem (Qiu and Wu, 2006)

Let f be a meromorphic function of lower order $\mu < \infty$ and deficiency $\delta(\infty, f) > 0$. Then,

$$m(L(f)) \geq \min\left(2\pi, \frac{4}{\mu} \arcsin\sqrt{\frac{\delta(\infty, f)}{2}}\right).$$

Quasiregular Maps

- We would like to generalize these types of results to higher dimensions.
- Quasiregular (qr) maps generalize analytic functions
  - qr maps send circles to ellipses with bounded distortion on small scales instead of circles to circles
  - there exists a constant $K \geq 1$ that bounds the distortion
  - qr maps are differentiable almost everywhere
  - qr maps are of transcendental-type if there is an essential singularity at ∞

Example

Let $f(x, y) = (2x, y)$. Then, f is 2-quasiregular.

Properties of Quasiregular Maps

Given an unbounded domain U of $\mathbb{R}^n$ and a function $f : U \to \mathbb{R}^n$, we define $M_U(r,f) = \sup_{|x|\le r, x\in U} |f(x)|$, as long as $\overline{B(0,r)}$ meets U.

If $U = \mathbb{R}^n$, we have $M(r,f) = \sup_{|x|=r} |f(x)|$.

Theorem (Bergweiler, 2006)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a quasiregular map. Then f is of transcendental-type if and only if

$$\lim_{r\to\infty} \frac{\log M(r,f)}{\log r} = \infty.$$

The order of an entire quasiregular map $f : \mathbb{R}^n \to \mathbb{R}^n$ is

$$\mu_f = \limsup_{r\to\infty} (n-1)\frac{\log\log M(r,f)}{\log r}.$$

Dynamics of Quasiregular Maps

Definition

Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a transcendental-type quasiregular map. The Julia set of f is defined to be

$$J(f) = \{x \in \mathbb{R}^n : \text{cap}\left(\mathbb{R}^n \setminus \cup_{k=1}^{\infty} f^k(U)\right) = 0\}$$

for every neighborhood U of x where cap is the conformal capacity of a condenser. The quasi-Fatou set is $QF(f) = \mathbb{R}^n \setminus J(f)$.

Julia Limiting Directions

- From Rickman's Theorem, we know $J(f)$ is unbounded for an entire transcendental-type qr map $f\colon \mathbb{R}^n \to \mathbb{R}^n$.

Definition

Let $n \geq 2$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a transcendental-type quasiregular map. We say $\zeta \in S^{n-1}$ is a Julia limiting direction of f if there exists a sequence $(x_m)_{m=1}^{\infty}$ in $J(f)$ with $\lim_{m\to\infty} |x_m| = \infty$ and $\lim_{m\to\infty} x_m/|x_m| = \zeta$.

As before, we denote the set of Julia limiting directions by $L(f)$.

Previous Results

It is not hard to see that $L(f)$ is a closed, non-empty subset of S^{n-1}.

Theorem (Fletcher, 2021)

Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a K-quasiregular map of transcendental-type. Suppose further that there exist $\alpha > 1$ and $\delta > 0$ such that for all large r there exists $s \in [r, \alpha r]$ such that

$$\min_{|x|=s} |f(x)| \geq \delta \max_{|x|=r} |f(x)|.$$

Then, $L(f) = S^{n-1}$.

Main Result

Denote the $(n-1)$-dimensional area of a subset E of S^{n-1} by $m(E)$ and the topological hull of U by $T(U)$.

Theorem (Fletcher and S., 2023)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a transcendental-type K-quasiregular map of order $\mu_f < \infty$ for which $T(U) \neq \mathbb{R}^n$ for any quasi-Fatou component U. Then there exists a component $E \subset L(f)$ with

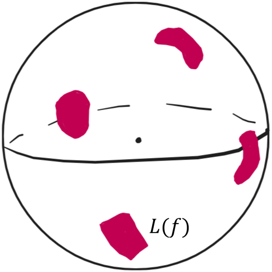
$$m(E) \geq \min(c_n K^{(2-2n)/n} \mu_f^{1-n}, \omega_{n-1})$$

where $c_n > 0$ is a constant depending only on n and $\omega_{n-1} = m(S^{n-1})$.

If $\mu_f \leq K^{-2/n} \left(\frac{c_n}{\omega_{n-1}}\right)^{1/(n-1)}$, then $L(f) = S^{n-1}$.

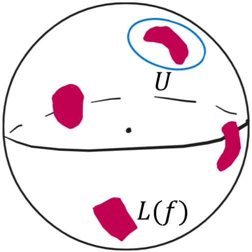
Proof Sketch

1. Suppose f is a transcendental-type qr map with order μ_f. Then, $L(f)$ is a closed non-empty subset of S^{n-1}. Suppose towards a contradiction that the largest component of $L(f)$ has measure less than $M = c_n K^{(2-2n)/n} \mu_f^{1-n}$.



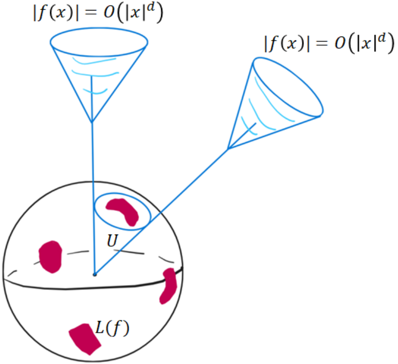
Proof Sketch

2. Let $\mathcal{F} = \{U : U$ is a domain in S^{n-1} such that $\partial U \subset S^{n-1} \backslash L(f)$ and $m(U) < M\}$. Note that $\mathcal{F}$ is an open cover of S^{n-1}.



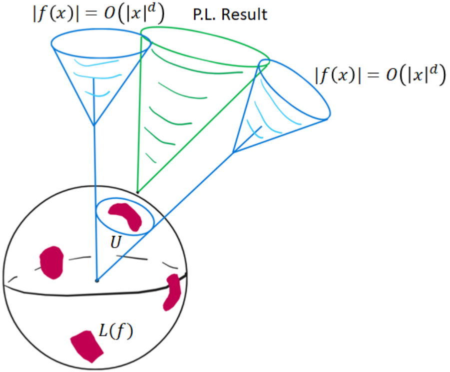
Proof Sketch

3. As notation, let $S_{x_0,E} = \{y = c(x - x_0) : c > 0, x \in E\}$. From a result by Fletcher, in sectors in $QF(f)$, we have $|f(x)| = O(|x|^d)$. Therefore, along $\partial S_{0,U}$, we have $|f(x)| = O(|x|^d)$.



Proof Sketch

4. Since $|f(x)| = O(|x|^d)$ along $\partial S_{0,U}$, we can use a Phragmén-Lindelöf result with the order μ_f to get a growth condition in U which contradicts f being of transcendental-type.



Phragmén-Lindelöf Principle

For transcendental entire functions $f : \mathbb{C} \to \mathbb{C}$, $\log|f|$ is subharmonic. Then we can use this result.

Theorem (Phragmén-Lindelöf Principle)

Let u be a subharmonic function in a sector $S = \{z \in \mathbb{C} : |\arg(z)| < \theta/2\}$ for $\theta \leq 2\pi$. Suppose $\limsup_{x \to y} u(x) \leq 0$ for all $y \in \partial S$. Then either $u \leq 0$ in S or $\liminf_{r \to \infty} M(r, u) r^{-\pi/\theta} > 0$ where $M(r, u) = \sup_{|z| \leq r} |u(z)|$.

- Let $u(z) = \log|f(z)|$. Then, this result gives us either $|f(z)| \leq 1$ in S or $\liminf_{r \to \infty} M(r, f) e^{r^{-\pi/\theta}} > 0$.
- Suppose $|f(z)| \leq C|z|^d$ on ∂S. Then, $\frac{|f(z)|}{C|z|^d} \leq 1$ on ∂S, so we get $|f(z)| \leq C|z|^d$ on S or $\liminf_{r \to \infty} M(r, f) e^{r^{-\pi/\theta}} > 0$.

Phragmén-Lindelöf Results for Quasiregular Maps

For a K-qr map $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $\log|f|$ is sub-F-extremal where the structural constants α and β can be written in terms of K.

Theorem (Granlund, Lindqvist, Martio, 1985)

Suppose that $S = S_{x_0,E}$ is a sector $\mathbb{R}^n$ and that F is a variational kernel of type n in S with structural constants α and β. Let $u\colon S \to \mathbb{R} \cup \{-\infty\}$ be a sub-F-extremal in S such that $\limsup_{x\to y} u(x) \le 0$ for $y \in \partial S$. Then either $u(x) \le 0$ in S or

$$\liminf_{r\to\infty} M_S(r,u) r^{-q} > 0$$

where $M_S(r,u) = \sup_{|x|\le r, x\in S} u(x)$, $q = d_n m(E)^{-1/(n-1)}(\alpha/\beta)^{1/n}$, and $d_n > 0$ is a constant depending only on n.

Phragmén-Lindelöf Results for Quasiregular Maps

Theorem (Fletcher and S., 2023)

Suppose that $S = S_{x_0,E}$ is a sector in $\mathbb{R}^n$ and that $F\colon S \times \mathbb{R}^n \to \mathbb{R}$ is a variational kernel of type n in S with structural constants α and β. Let $u\colon S \to \mathbb{R} \cup \{-\infty\}$ be a sub-F-extremal in S such that $\limsup_{x \to y} u(x) \leq C \log^+ |y|$ at each $y \in \partial S$ for some constant $C > 0$. Then given $\varphi > 0$, either for all sufficiently large $|x|$ in S we have

$$u(x) \leq C(1+\varphi) \log |x|$$

or

$$\limsup_{r \to \infty} M_S(r,u) r^{-D} > 0$$

where $D = d_n m(E)^{-1/(n-1)} (\alpha/\beta)^{1/n} \varphi/(1+\varphi)$ and $d_n > 0$ depends only on n.

Inverse Problem

Question

Given a set $E \subset S^{n-1}$, can we find a quasiregular map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $L(f) = E$?

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Theorem (Nicks and Sixsmith, 2018)

There exists a quasiregular map of transcendental-type $g : \mathbb{R}^3 \to \mathbb{R}^3$ that is equal to the identity map in a half-space.

By considering quasiconformal conjugates, Fletcher gave a partial answer to the inverse problem in $\mathbb{R}^3$: when E consists of the union of the closures of finitely many domains in S^2, there exists a quasiregular map of finite lower order such that $E = L(f)$.

Inverse Problem

To solve the inverse problem, we would like to construct a qr map with one Julia limiting direction similar to the construction by Nicks and Sixsmith.

As notation, let $H(x_0, \theta, w) \subset \mathbb{R}^n$ be a half-beam with width $2w > 0$ along the center ray $R = \{x \in \mathbb{R}^n : \frac{x - x_0}{|x - x_0|} = \theta\}$ starting at $x_0 \in \mathbb{R}^n$ extending in the $\theta \in S^{n-1}$ direction.

Inverse Problem

Theorem (Fletcher and S., 2023)

Let $E \subset S^{n-1}$ be closed. Suppose that there exists a quasiregular map F that satisfies

1. $L(F) = \{e_1\}$
2. $F(x) = x$ *for* $x \in \mathbb{R}^n \backslash H(0, e_1, 1)$.

Then, there exists a quasiregular map f with $L(f) = E$.

Theorem (Fletcher and S., 2023)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a transcendental-type K-quasiregular map for which $T(U) \neq \mathbb{R}^n$ for any quasi-Fatou component U and $L(f) = \{x_0\}$. Then, f has infinite order.

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Thank you!