Planar Quasiregular Linearization

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Linearization Theorems in the Plane

Suppose f is holomorphic at $z_0 = 0$ with $f(0) = 0$ and $f'(0) = \lambda$:

$$f(z) = \lambda z + \sum_{n=2}^{\infty} a_n z^n \text{ then}$$

Theorem (König's and Böttcher's Theorems)

In a neighborhood of 0 *there is a holomorphic function* ϕ *so that*

- *König:* $\phi \circ f(z) = \lambda \phi(z)$ *when* $|\lambda| \neq 0, 1$ *or*
- *Böttcher:* $\phi \circ f(z) = (\phi(z))^n$ *when* $\lambda = 0$.

Generalizing to quasiregular maps

Definition

Let f be a qr map in a neighborhood of $z_0 \in \mathbb{C}$ with $f(z_0) = z_0$.

(a) We say z_0 is a geometrically attracting fixed point if there exist $\lambda \in (0, 1)$ and a neighborhood U of z_0 such that

$$|f(z) - z_0| \leq \lambda |z - z_0| \quad \text{for } z \in U. \tag{1}$$

(b) If for every $\lambda \in (0, 1)$ there is a nbhd U_λ such that (1) holds for $z \in U_\lambda$, then z_0 is a geometrically superattracting fixed point.

Generalizing to quasiregular maps

Let f be a qr map and 0 a geometrically (super)attracting fixed point.

Question:

Can we determine a suitable map h and a quasiconformal map ψ so that $\psi \circ f = h \circ \psi$?

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- Hinkkanen-Martin-Mayer (2004) shows that we can "linearize" f to a generalized derivative when f is uniformly quasiregular.
- Fletcher-Fryer (2012), have a version of a Böttcher's coordinate for quasiregular mappings in the plane with constant complex dilatation.

Definition of generalized derivative

Definition (Gutlyanskii, Martio, Ryazanov, Vuorinen, (2000))

Let $f : D \to \mathbb{R}^n$ be a qr map, $D \subset \mathbb{R}^n$, and $x_0 \in D$.
A generalized derivative φ of f at x_0 is defined by the following limit

$$\varphi(x) = \lim_{k\to\infty} \frac{f(x_0 + t_k x) - f(x_0)}{\rho_f(t_k)} \quad \text{(when it exists)} \tag{2}$$

where $(t_k)_{k=1}^{\infty} \to 0,$ and each $t_k > 0$.

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Here, $\rho_f(t_k)$ is the mean radius of the image of $B(x_0, t_k)$ given by

$$\rho_f(t_k) = \left(\frac{\mu[f(B(x_0, t_k))]}{\mu[B(0, 1)]}\right)^{1/n}, \tag{3}$$

where μ is the standard Lebesgue measure.

Generalized derivative example 1

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping.
Since f is a linear mapping

$$\mu(f(B(x_0, t))) = t^n \mu(B(0, 1)) J_f(x_0).$$

At x_0 we have

$$\rho_f(t) = t J_f(x_0)^{1/n}.$$

The generalized derivative of f at x_0 is

$$\lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t J_f(x_0)^{1/n}} = \frac{f'(x_0)x}{J_f(x_0)^{1/n}}.$$

Generalized derivative example 2

Let $f(z) = z^d$, for $z \in \mathbb{C}$ with $0 < d \in \mathbb{Z}$. At $z_0 = 0$, we have

$$\rho_f(t) = \left(\frac{\mu(f(B(0,t)))}{\mu(B(0,1))} \right)^{1/2} = \left(\frac{\pi (t^d)^2}{\pi} \right)^{1/2} = t^d.$$

The generalized derivative of f at 0 is

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Note:

- Under the gen. deriv., we lose information on scaling, but maintain information about the shape of f near x_0.
- The set of gen. deriv. of f at x_0 is the infinitesimal space of f at x_0, denoted by $T(x_0, f)$.
- We call $T(x_0, f)$ simple if there is only one gen. deriv. of f at x_0.

Properties of simple quasiregular mappings

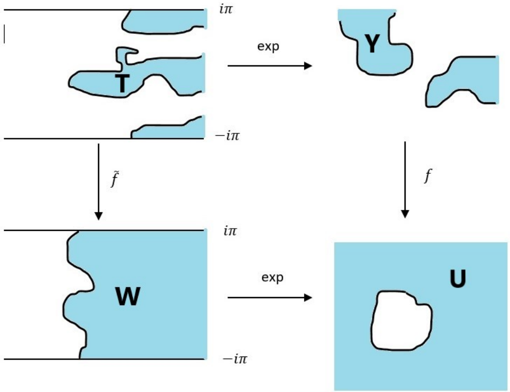
Theorem (Gutlyanskii, Martio, Ryazanov, Vuorinen, (2000))

Let $f : U \to \mathbb{R}^n$, $n \geq 2$, be a non-constant qr map and $0 \in U$. If $T(0, f)$ has only $g : \mathbb{R}^n \to \mathbb{R}^n$, with $f(0) = 0 \in U$, then f has rep.

$$f(x) \sim \rho_f(|x|) g\left(\frac{x}{|x|}\right) := \mathcal{D}(x). \quad (4)$$

Note: Our candidate for h is $\mathcal{D}$, if $\mathcal{D}$ is quasiregular.

Logarithmic transform



The blue region in the image of f is a Jordan domain not containing singularities and $\{0, f(0)\}$.

When $\mathcal{D}$ is not quasiconformal

Proposition (Fletcher-P (2023))

Suppose that $e^M > 0$ *and let* $f : B(0, e^M) \to \mathbb{R}^2$ *be a qc map that fixes* 0 *and for some* $0 < e^{t_0} < e^M$*, the image of the circle centred at* 0 *of radius* e^{t_0} *under* f *is a non-rectifiable curve.*

Then $\widetilde{\rho}_f$ *is not bi-Lipschitz at* t_0*.*

$$\textbf{Recall: } \mathcal{D}(x) = \rho_f(|x|)g\left(\frac{x}{|x|}\right)$$

BIP condition for the plane

Definition

Let $R > 0$ and let $f : B(0, R) \to \mathbb{C}$ be qr with $f(0) = 0$.

We say that f is a bounded integrable parameterization map if there exists $P > 0$ such that for every $t < \log R$, the parameterization $\gamma_t : [-\pi, \pi] \to \mathbb{C}$ defined by $\gamma_t(s) = \widetilde{f}(t + is)$ satisfies

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Note: that for the case $f(z) = z^d$ we have $\widetilde{f}(z) = dz$ and so

$$\int_{-\pi}^{\pi} |\gamma_t'(s)|^2 \, ds = 2\pi d^2.$$

In particular, P may depend on $i(0, f)$.

Okay, so $\mathcal{D}$ can be quasiconformal

Theorem (Fletcher-P (2023))

Let $K \geq 1$, $R > 0$ and let $f : B(0, R) \to \mathbb{C}$ be a simple BIP K-quasiregular map with $f(0) = 0$.

Then there exists L depending only on K and P such that

- $\widetilde{\rho}_f$ *is L-bi-Lipschitz on* $(-\infty, \log R)$ *and*
- $\widetilde{\mathcal{D}}$ *is L-bi-Lipschitz for* $Re(z) < \log R$.

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In particular, it follows from this result that $\mathcal{D}$ itself is quasiregular on a neighbourhood of 0.

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- We will look at logarithmic transform of the ψ_i.
- We need to show that the sequence is:
  - uniformly convergent
  - there is a uniform bound on the maximal dilatations.

Needed Conditions and assumptions

Note we will need:

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Ca) A uniform estimate on how close the comp.deriv. of $\tilde{f}$ and $\widetilde{\mathcal{D}}$ are.

Cb) A uniform estimate on how close $\tilde{f}$ and $\widetilde{\mathcal{D}}$ are that guarantees:

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D) In the König's case, we need to bound λ to counteract the L-Bi-Lipschitz constant from $\widetilde{\mathcal{D}}$.

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2. How unique is ψ?
3. For quasiregular maps in three or higher dimensions, can we obtain a similar result even though the set of branch points is no longer discrete?

The end

Thank You!

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