Dynamics of quasiregular maps

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Introduction

- This talk will survey some research aiming to generalise complex dynamics to quasiregular mappings on $\mathbb{R}^d$.

- Quasiregular maps were introduced in the mid-1960s and many results from classical function theory have quasiregular counterparts.

- We will discuss analogues of both polynomial / rational dynamics on $\mathbb{C} \cup \{\infty\}$ and transcendental dynamics on $\mathbb{C}$.

- We will mention the special case of *uniformly* quasiregular dynamics, but we will often work without this restriction.

Quasiregular mappings

Definition

A continuous $f \colon \mathbb{R}^d \to \mathbb{R}^d$ is *quasiregular* (qr) if $f \in W^1_{d,\mathrm{loc}}(\mathbb{R}^d)$ and there exists $K_O \geq 1$ such that

$$\|Df(x)\|^d \leq K_O J_f(x) \qquad \text{a.e.}$$

where $\|Df(x)\|$ is the norm of the derivative and $J_f(x)$ is the Jacobian.

- Informally, a qr map sends infinitesimal spheres to infinitesimal ellipsoids of bounded eccentricity.
- A mapping is called K-qr if the local distortion is $\leq K$.
- A *quasiconformal* map is simply an injective quasiregular map.
- Non-constant qr maps are open, discrete and almost everywhere differentiable (Reshetnyak, 1960s).

Simple examples

- $(x, y) \mapsto (Kx, y)$ is K-qr on $\mathbb{R}^2$.
- On $\mathbb{R}^d$, $x \mapsto |x|^\alpha x$ is qr for $\alpha > -1$.
- Holomorphic functions on $\mathbb{C}$ are 1-qr.
- With $N \in \mathbb{N}$ and cyl. co-ords on $\mathbb{R}^3$, the map $(r, \theta, x_3) \mapsto (r, N\theta, x_3)$.

- A non-constant qr map $f : \mathbb{R}^d \to \mathbb{R}^d$ is called *polynomial type* if $|f(x)| \to \infty$ as $|x| \to \infty$. Otherwise, f is *transcendental type*.

- f is polynomial type $\iff$ $\deg(f) < \infty$.

- Can also consider quasiregular self-maps of $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ that are analogous to rational functions on $\overline{\mathbb{C}}$.

- If f, g are quasiregular, then $f + g$ need **not** be qr, but $f \circ g$ **is** qr.
- If f is K-qr, then the nth iterate f^n is only K^n-qr in general. $(\star)$

There are quasiregular analogues of Picard's and Montel's theorems:

Theorem (Rickman)

There exists a constant $q = q(d, K)$ with the following property: every K-qr map $f : \mathbb{R}^d \to \mathbb{R}^d$ that omits q values must be constant.

Theorem (Miniowitz)

Let $\mathcal{F}$ be a family of K-qr maps on a domain $D \subset \mathbb{R}^d$.
If there exist distinct points $a_1, \dots, a_q$ that are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.

$(\star)$ means we can't apply "Montel" to family of iterates $\{f^n\}$ in general.

Uniformly quasiregular maps

- If every iterate f^n is K-quasiregular with the same K, then f is called *uniformly quasiregular* (uqr).
- For uqr maps, the usual definition of Fatou and Julia sets via normality works well.

Theorem (Hinkkanen, Martin, Mayer)

For a non-injective uniformly quasiregular map $f : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$

- $J(f^p) = J(f)$;
- $J(f)$ *is perfect;*
- *classification of periodic points and periodic Fatou components;*
- $J(f) =$ *boundary of any attracting basins.*

Blowing-up: If an open set U meets $J(f)$ then Miniowitz's Montel theorem gives that $\overline{\mathbb{R}^d} \setminus O^+(U)$ contains $\leq q$ points.

Two open questions on uqr maps

(1) Do there exist uqr maps of transcendental type in dim ≥ 3?

(2) For uqr maps on $\overline{\mathbb{R}^d}$, is the Julia set the closure of the repelling periodic points?

- Siebert showed that every Julia point is a limit of periodic points, and moreover that the answer to (2) is "yes" if $J(f) \not\subset \overline{\{\text{post-critical set}\}}$.
- Fletcher and Stoertz showed (2) is also "yes" for polynomial type uqr maps $\mathbb{R}^d \to \mathbb{R}^d$ for which $J(f)$ is a Cantor set.
- There are examples of uqr maps where $J(f)$ is a wild Cantor set — see Vyron Vellis talk.

Examples of qr maps

1. The Zorich map $Z\colon \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$ is a qr (not uqr) analogue of the exponential function.
   - It is periodic in the x_1 and x_2 directions (so is transcendental type).
   - Exponential growth/decay in x_3 direction: $|Z(x_1, x_2, x_3)| = e^{x_3}$. (Note Z maps the plane $\{x_3 = C\}$ onto the sphere $\{|x| = e^C\}$.
2. By interpolating, we can construct a qr map on $\mathbb{R}^3$ that is the identity on $\{x_3 \le 0\}$ and is like $Z + \mathrm{id}$ on $\{x_3 \ge L\}$.
3. Let $d \in \mathbb{N}$. We can define a uniformly qr power map $P_d\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$P_d(x) = Z(dZ^{-1}(x)).$$

   (Compare with $z^d = \exp(d \log z)$ for $z \in \mathbb{C}$.)
   Then $|P_d(x)| = |x|^d$ and so the Julia set of P_d is the unit sphere.
4. Zorich and power maps work in any dimension $\mathbb{R}^n \to \mathbb{R}^n$.

Non-uniformly quasiregular dynamics

Now let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ or $\overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ be K-qr, but <u>not</u> assumed uqr. So we can't apply the qr version of Montel's theorem to $\{f^n\}$.

Extending an idea of Sun and Yang we use a 'blowing-up' property to *define* the Julia set:

$$J(f) := \left\{ x : \text{for every nhd } U \text{ of } x,\ \mathbb{R}^d \backslash O^+(U) \text{ is small} \right\}.$$

Here "small" means conformal capacity zero.
It follows immediately that $J(f)$ is closed and completely invariant.

Theorem (Bergweiler (deg $< \infty$), Bergweiler + N. (deg $= \infty$))

The definition of $J(f)$ above agrees with the usual one if f is uqr.
If $\deg(f) > K_I$, then $J(f) \neq \emptyset$ and, in fact, $J(f)$ is infinite.

Assume $f\colon \mathbb{R}^d \to \mathbb{R}^d$ or $\overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ is K-qr, with $\deg(f) > K_I$. Recall

$$J(f) := \left\{ x : \text{for every nhd } U \text{ of } x,\ \mathbb{R}^d \backslash O^+(U) \text{ is small} \right\}.$$

Conjecture

- Equivalent to replace "small" by "finite" in $J(f)$ definition.
- $J(f)$ is perfect and $J(f^p) = J(f)$.

The conjecture is open in general, but holds under a variety of extra hypotheses. In particular, it holds if

- dimension $= 2$; or
- f is locally Lipschitz; or
- f is trans type and bounded on some path to ∞; or
- f is trans type and has bounded local index.

Examples of Julia sets

- For Zorich map Z on $\mathbb{R}^3$ and sufficiently small $\lambda > 0$, the Julia set $J(\lambda Z)$ is a Cantor bouquet.
  It is the complement of the attracting basin of the fixed point of λZ.
  [Bergweiler, (+N.)]

- For a modified Zorich map $\tilde{Z}$ and sufficiently large $\lambda > 0$, the Julia set $J(\lambda \tilde{Z}) = \mathbb{R}^3$. [Tsantaris]

The quasi-Fatou set

For a general quasiregular map f, we call the complement of the Julia set the *quasi-Fatou set*:

$$\mathcal{QF}(f) := \mathbb{R}^d \setminus J(f).$$

Note: no normality assumption!

Behaviour in $\mathcal{QF}(f)$ can be quite different to that in Fatou components in complex dynamics. Inside a single quasi-Fatou component we can have:

- several distinct attracting fixed points;
- some points iterating to ∞ while others have bounded orbits.

Escaping sets

Let f be trans entire on $\mathbb{C}$ or trans type qr on $\mathbb{R}^d$. We define

- the *escaping set* $I(f) = \{x : f^n(x) \to \infty\}$;
- the *fast escaping set* $A(f) \subset I(f)$, roughly points escaping faster than an iterated maximum modulus $M^n(r, f)$;
- the *bounded orbit set* $BO(f) = \{x : (f^n(x))_{n \geq 0} \text{ is bounded}\}$.

In complex dynamics,

$$J(f) = \partial I(f) = \partial A(f) = \partial BO(f).$$

In qr dynamics, $J(f) \subset \partial I(f) \cap \partial BO(f)$ but inclusion can be strict.

Theorem (Bergweiler, Fletcher, N.)

If f is qr such that $\liminf_{r \to \infty} \dfrac{\log \log M(r, f)}{\log \log r} = \infty$, *then* $J(f) = \partial A(f)$.

Remark $J(f) = \partial A(f)$ gives a 'normality' property: if one point in some quasi-Fatou component is fast-escaping, then every point is.

Types of quasi-Fatou component

We can describe quasi-Fatou components as invariant ($f(U) \subset U$), periodic ($f^p(U) \subset U$), pre-periodic or wandering.

We say that a domain $U \subset \mathbb{R}^d$ is *full* if its complement has no bounded components; otherwise it's *hollow*. (cf. simply/multiply-conn in the plane.)

Points in full periodic quasi-Fatou components cannot be fast-escaping:

Theorem (N., Sixsmith)

If f is trans type qr on $\mathbb{R}^d$ and U is a full component of $\mathcal{QF}(f)$ such that $f^p(U) \subset U$ then, for $x \in U$,

$$\log\log|f^{pn}(x)| = O(n) \quad \text{as } n \to \infty.$$

In particular, $U \cap A(f) = \emptyset$.

The above is analogous to Baker domains in transcendental complex dynamics (except here we necessarily have log log rather than just log.)

Hollow quasi-Fatou components

Baker showed that for trans entire f on $\mathbb{C}$, all multiply-connected Fatou components are bounded (and hence wandering).

Now let f be trans type qr on $\mathbb{R}^d$.

Conjecture

All hollow quasi-Fatou components are bounded (and hence wandering).

Theorem (N., Sixsmith)

If the conjecture holds then $J(f) = \partial A(f)$, and $J(f) = J(f^p)$ for $p \in \mathbb{N}$, and $J(f)$ is perfect.

Hollow quasi-Fatou components

Bounded hollow quasi-Fatou components of qr maps behave just like multiply-connected Fatou components in complex dynamics.

Theorem (N., Sixsmith)

Let U_0 be a bounded hollow component of $\mathcal{QF}(f)$. Then U_0 is wandering and

- $U_n := f^n(U_0)$ *is a bounded hollow component of* $\mathcal{QF}(f)$ *for all* n*;*
- U_{n+1} *surrounds* U_n *for all large* n*;*
- $\overline{U_n} \subset A(f)$*;*
- *the 'inner' and 'outer' boundaries of* U_n *are far apart for large* n*.*

Existence of hollow quasi-Fatou components

Theorem (N., Sixsmith)

For each $d \geq 2$, there exists a quasiregular f on $\mathbb{R}^d$ of trans type such that $\mathcal{QF}(f)$ has a hollow component.

In that construction we have a poor grasp on $J(f)$. We have either

- a sequence of wandering bounded hollow $\mathcal{QF}$ components; or
- a hollow $\mathcal{QF}$ component that is unbounded.

But we don't know which.

Recent work with Burkart and Fletcher interpolates between power maps of increasing degree to construct a trans type qr map of $\mathbb{R}^3$ with bounded hollow quasi-Fatou components separated by spheres $\{|x| = r_k\}$ in $J(f)$.

I believe this also gives the first example in dim ≥ 3 where the quasi-Fatou set has more than two components. See next talk!