Topological obstructions for uniformly quasiregular dynamics

Ilmari Kangasniemi

University of Cincinnati

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Quasiregular maps

Let M, N be oriented Riemannian n-manifolds, and let f be a continuous map in the local Sobolev space $W^{1,n}_{\text{loc}}(M, N)$.

The map f is called *K-quasiregular* (K-QR) for $K \geq 1$ if f satisfies

$$|Df(x)|^n \leq K J_f(x)$$

for almost every $x \in M$, where:

- $Df : TM \to TN$ is the weak derivative;
- $|\cdot|$ is the operator norm with respect to the Riemannian metrics;
- J_f is the Jacobian determinant induced by the orientations.

A map is called *quasiregular* (QR) if it is K-QR for some $K \geq 1$. Homeomorphic quasiregular maps are called *quasiconformal*.

Uniformly quasiregular maps

Let M be an oriented Riemannian n-manifold, and let $f\colon M \to M$ be a quasiregular self-map.

The map f is called *uniformly K-quasiregular* (K-UQR) for $K \geq 1$ if every iterate f^k of f is K-quasiregular.

- UQR maps share many properties with holomorphic dynamics:
  - A UQR map f has a Julia set $\mathcal{J}_f$ and a Fatou set $\mathcal{F}_f$.
  - (Okuyama–Pankka) A UQR map f has an invariant, balanced probability measure μ_f supported on $\mathcal{J}_f$.
  - (Iwaniec–Martin) A UQR map f is 1-UQR with respect to a Riemannian metric g_f on M — but this g_f is really non-smooth, one only has $g_f \in L^\infty(M, T^*M \otimes T^*M)$ with no differentiability.
- Compared to QR, UQR is a far more restrictive condition. It is hard to come up with examples.

Motivating question

Question

Which closed (compact and boundaryless), connected Riemann surfaces admit non-constant non-injective holomorphic self-maps?

- $\mathbb{S}^2$ admits branching ones.
- $\mathbb{T}^2$ admits locally injective ones.
- All the rest are hyperbolic, which doesn't allow for any.

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- All the rest are hyperbolic, which doesn't allow for any.

Question

Which closed, connected, orientable n-manifolds M admit non-constant non-injective UQR self-maps?

- A much more complicated question, that is far from being resolved.

Existing examples for $n \geq 3$

- Conformal trap -constructions.
  - Involve creating a "trap" that is mapped into itself conformally (or uniformly quasiregularly).
  - Original example by Iwaniec and Martin.
  - So far only successful in spaces universally covered by $\mathbb{S}^n$ ($\mathbb{S}^n$ itself, projective space, lens spaces, etc.)
- Lattès-type maps.
  - $f\colon M \to M$ given by $f \circ h = h \circ A$, where $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear conformal map, and $h\colon \mathbb{R}^n \to M$ is a quasiregular map that is strongly automorphic under a discrete isometric groups action on $\mathbb{R}^n$.
  - Either M, $M \setminus \{x_0\}$, or $M \setminus (\{x_0, x_1\})$ must be homeomorphic to a quotient of $\mathbb{R}^n$ for such an example to be possible.
  - Examples hence are most typical on torus-like spaces such as $\mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_l}$, where $k_1 + \cdots + k_l = n$.

Low dimensions

- $n = 2$: the Riemann surfaces that admit non-constant non-injective UQR maps are exactly the ones that admit non-constant non-injective holomorphic self-maps.
- $n = 3$ is the last dimension with a full characterization of which topological 3-manifolds admit non-constant non-injective UQR dynamics. This is thanks to Perelman's proof of Thurston's geometrization conjecture.

  The full characterization in 3 dimensions was assembled in the PhD thesis of Kangaslampi.
- From $n = 4$ onwards, the question is open.

Restrictions that do not use the "Uniformly" -part

Hyperbolic closed manifolds can also be excluded in higher dimension, under the following precise theorem.

Theorem (Bridson-Hinkkanen-Martin)

If a closed, connected, oriented Riemannian n-manifold M has a torsion-free, non-elementary, word-hyperbolic fundamental group, then every non-constant QR-map $f : M \to M$ is a homeomorphism.

Connection to quasiregular ellipticity

A closed, connected, oriented Riemannian n-manifold M is called *quasiregularly elliptic* (QR-elliptic) if there exists a non-constant quasiregular map $h\colon \mathbb{R}^n \to M$.

The following fact is proven in the thesis of Kangaslampi.

Theorem

If a closed, connected, oriented Riemannian n-manifold admits a non-constant non-injective uniformly quasiregular self-map, then it is quasiregularly elliptic.

Thus, it makes sense to say that a manifold is *uniformly quasiregularly elliptic* (UQR-elliptic) if it admits a non-constant non-injective uniformly K-quasiregular self-map.

Consequence of QR-ellipticity

Theorem (Prywes)

If a closed, connected, oriented Riemannian n-manifold is M QR-elliptic, then for all $k \in \{0, \dots, n\}$, its k:th cohomology group satisfies

$$\dim H^k(M; \mathbb{R}) \leq \binom{n}{k}$$

This improves to

Theorem (Heikkilä-Pankka)

If a closed, connected, oriented Riemannian n-manifold M is QR-elliptic, then there is an embedding of graded algebras $H^*(M; \mathbb{R}) \hookrightarrow \wedge^* \mathbb{R}^n$, where the target is the standard n-dimensional exterior algebra.

Are QR-elliptic manifolds always UQR-elliptic?

No, they are not.

Theorem (Rickman)

The manifold $(\mathbb{S}^2 \times \mathbb{S}^2)\#(\mathbb{S}^2 \times \mathbb{S}^2)$ *is QR-elliptic.*

Theorem (Kangasniemi)

The manifold $(\mathbb{S}^2 \times \mathbb{S}^2)\#(\mathbb{S}^2 \times \mathbb{S}^2)$ *is not UQR-elliptic.*

The difference is:

Theorem (Kangasniemi)

If a closed, connected, oriented Riemannian n-manifold M is UQR-elliptic, then the embedding $H^*(M;\mathbb{R}) \hookrightarrow \wedge^*\mathbb{R}^n$ *can be selected so that its image is closed under the standard Clifford product of* $\wedge^*\mathbb{R}^n$.

Restrictions on degree

Theorem (Kangasniemi-Pankka)

If M is a closed, connected, oriented Riemannian n-manifold and $f\colon M \to M$ is a non-constant UQR-map, then

$$(\deg f)^{\frac{k}{n} \dim H^k(M;\mathbb{R})} \in \mathbb{Z}$$

for all $k \in \{1, \ldots, n-1\}$.

The above is a clear difference between QR and UQR maps. For instance, if $M = \mathbb{S}^{n-1} \times \mathbb{S}^1$ with $n \geq 3$,

- QR-maps $f\colon M \to M$ can have any positive degree, since the map $(x, e^{i\theta}) \mapsto (x, e^{mi\theta})$ is QR with $K = m^{n-1}$.
- UQR-maps $f\colon M \to M$ can only have positive degrees of the form $m^n, m \in \mathbb{Z}_{>0}$ by the above result.

Julia sets

Question (Martin–Mayer)

If $f : \mathbb{S}^n \to \mathbb{S}^n$, $n \geq 3$, is a non-constant non-injective UQR map with a Julia set $\mathcal{J}_f$ of positive measure, is f a Lattès-type map?

They manage a special case of this.

Theorem (Martin–Mayer)

If $f : \mathbb{S}^n \to \mathbb{S}^n$, $n \geq 3$, is a non-constant non-injective UQR map with a set of conical points $\Lambda_c(f)$ of positive measure, then f is a Lattès-type map.

For rational maps in $\mathbb{S}^2$, $\Lambda_c(f)$ is large in $\mathcal{J}_f$, in the sense that $\mu_f(\Lambda_c(f)) = 1$. However, whether this is true for UQR-maps when $n \geq 3$ is open.

Julia sets

Theorem (Kangasniemi)

If M is a closed, connected, oriented Riemannian n-manifold, and if $H^*(M;\mathbb{R}) \ncong H^*(\mathbb{S}^n;\mathbb{R})$, then every non-injective non-constant UQR self-map on M has a Julia set of positive measure, with μ_f absolutely continuous with respect to the Riemannian volume.

On $\mathbb{S}^n$ one has multiple examples of UQR-maps that have a Julia set of zero measure. But by the above result, if one moves to a closed manifold M with nontrivial cohomology, UQR maps are too strongly restricted for this to happen.

Julia sets

Thus, combining the previous result with the question of Martin and Mayer, we encounter the following open question.

Question

If M is a closed, connected, oriented Riemannian n-manifold, and if $H^*(M;\mathbb{R}) \not\cong H^*(\mathbb{S}^n;\mathbb{R})$, then is every non-constant non-injective UQR self-map $f\colon M \to M$ a Lattès map?

Even just the following simpler question is open.

Question

If M is a closed, connected, oriented Riemannian n-manifold, and if $H^*(M;\mathbb{R}) \not\cong H^*(\mathbb{S}^n;\mathbb{R})$, then does every non-constant non-injective UQR self-map $f\colon M \to M$ have $\mathcal{J}_f = M$?

Thank you for your attention.