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Moments and positivity

Aimo Hinkkanen

University of Illinois Urbana–Champaign

aimo@illinois.edu

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Plan of the talk

– Definition of moments

– Review of parts of classical theory of moments

– Uniqueness criteria for signed measures

– Uniqueness criteria for real and complex functions

– Positivity criteria for signed measures

– Positivity criteria for functions

– Positivity criteria for polynomials

– Examples of positive polynomials

– Criteria for polynomials to have only real zeros

– Criteria for some entire functions to have only real zeros

Signed measures

We assume that all **signed measures** are Borel measures, not the zero measure, and not concentrated at the origin.

If ν is a signed measure on $\mathbb{R}$, then ν has a Jordan decomposition

$$\nu = \nu^+ - \nu^-,$$

where ν^+ and ν^- are mutually singular non-negative measures. We set

$$|\nu| = \nu^+ + \nu^-.$$

If ν is a measure in the usual sense, that is, non-negative, then $\nu = \nu^+ = |\nu|$ and $\nu^- = 0$.

Moments and absolute moments

For integers $n \geq 0$, we define the **moments** $s_n(\nu)$ and **absolute moments** $s_n^*(\nu)$ of ν by

$$s_n(\nu) = \int_{-\infty}^{\infty} x^n \, d\nu(x), \quad s_n^*(\nu) = \int_{-\infty}^{\infty} |x|^n \, d|\nu|(x),$$

if these integrals exist and are finite. We have $s_n(\nu) = s_n(\nu^+) - s_n(\nu^-)$.

By our assumptions, $s_n^*(\nu) > 0$ for all $n \geq 0$. We have

$$s_n(\nu) \leq |s_n(\nu)| \leq s_n^*(\nu)$$

for all $n \geq 0$.

If $\nu \geq 0$, then $\nu = |\nu|$, so

$$s_{2n}^*(\nu) = s_{2n}(\nu)$$

for all $n \geq 0$.

Moments and absolute moments of functions

The signed measure ν may be of the form $d\nu(x) = f(x)\,dx$ for a function f. If $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, then for $n \geq 0$, we define the moments $s_n(f)$ and absolute moments $s_n^*(f)$ of f by

$$s_n(f) = \int_{-\infty}^{\infty} x^n f(x)\,dx, \quad s_n^*(f) = \int_{-\infty}^{\infty} |x|^n |f(x)|\,dx \tag{1}$$

if these integrals exist and are finite.

For every Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ there is a Borel function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ for a.e. $x \in \mathbb{R}$. The functions f and g have the same moments, and the same absolute moments. Replacing f by g, if necessary, we get a Borel measure $d\nu(x) = g(x)\,dx$.

We assume that f does not vanish almost everywhere. Then $s_n^*(f) > 0$ for all $n \geq 0$.

Moment problems

More generally, if K is a closed subset of $\mathbb{R}^d$ and μ is a measure on K, then with the notation $x = (x_1, x_2, \ldots, x_d) \in K$, we can define the moments

$$s_{n_1,n_2,\ldots,n_d}(\nu) = \int_K x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d} \, d\nu(x)$$

for integers $n_1, n_2, \ldots, n_d \geq 0$ if the integrals exist.

Take $d = 1$, $K \subset \mathbb{R}^d = \mathbb{R}$. Suppose that real numbers μ_n for $n \geq 0$ are given. We ask for which sequences μ_n there is a measure ν such that for all $n \geq 0$ we have $s_n(\nu) = \mu_n$ (and if so, whether ν is unique). This is a **moment problem**.

$K = \mathbb{R}$: Hamburger moment problem (1921)

$K = [0, \infty)$: Stieltjes moment problem (1885)

$K = [0, 1]$: Hausdorff moment problem (1923)

$K \subset \mathbb{R}^d$ closed: Schmüdgen (1979) and others

Applications of moment problems

Monograph: Moments, Positive Polynomials and Their Applications, by Jean Bernard Lasserre, Imperial College Press, 2010.

– Finding the global minimum of a function on a subset of $\mathbb{R}^n$
– Solving equations
– Computing the Lebesgue volume of a subset $S \subset \mathbb{R}^n$
– Computing an upper bound on $\mu(S)$ over all measures μ satisfying some moment conditions
– Pricing exotic options in Mathematical Finance
– Computing the optimal value of an optimal control problem
– Evaluating an ergodic criterion associated with a Markov chain
– Evaluating a class of multivariate integrals
– Computing Nash equilibria
– With $\hat{f}$ the convex envelope of a function f, evaluate $\hat{f}(x)$ at some given point x

Moments of positive measures on $\mathbb{R}$, necessary condition

If ν is a non-negative measure on $\mathbb{R}$, then for all real numbers $c_0, c_1, \ldots, c_k$ we have

$$0 \leq \int_{-\infty}^{\infty} \left(\sum_{j=0}^{k} c_j x^j \right)^2 d\nu(x),$$

which can be written as

$$\sum_{i=0}^{k} \sum_{j=0}^{k} c_i c_j s_{i+j}(\nu) \geq 0.$$

Thus, for each $k \geq 0$, the $(k+1) \times (k+1)$-matrix with (i,j)-entry $s_{i+j}(\nu)$ for $0 \leq i,j \leq k$, is positive definite.

Moments of positive measures on $\mathbb{R}$, necessary condition in terms of determinants

If $\nu \geq 0$ and ν is not concentrated at finitely many points, then if not all c_i vanish, we have

$$\sum_{i=0}^{k}\sum_{j=0}^{k} c_i c_j s_{i+j}(\nu) = \int_{-\infty}^{\infty} \left(\sum_{j=0}^{k} c_j x^j \right)^2 d\nu(x) > 0.$$

Thus the matrix with entries $s_{i+j}(\nu)$, $0 \leq i, j \leq k$, is strictly positive definite, so that if Δ_k denotes its **determinant**, then

$$\Delta_j > 0$$

for $0 \leq j \leq k$ (the leading principal minors are of the same form), hence for all $j \geq 0$. There are more refined results for measures supported on a finite set.

Moments of positive measures, sufficient condition

Suppose that the real numbers μ_n for $n \geq 0$ satisfy

$$\sum_{i=0}^{k} \sum_{j=0}^{k} c_i c_j \mu_{i+j} \geq 0$$

in all cases. Then there exists a measure $\nu \geq 0$ such that

$$s_n(\nu) = \mu_n$$

for all $n \geq 0$.

The proof is based on the **Riesz representation theorem**.

Moments of signed measures

R.P. Boas, Jr., The Stieltjes moment problem for functions of bounded variation, Bull. Amer. Math. Soc. 45 (1939), 399–404:

For every sequence μ_n of real numbers there is a signed measure ν on $[0, \infty)$ such that

$$s_n(\nu) = \mu_n$$

for all $n \geq 0$.

Taking $\nu = 0$ on $(-\infty, 0)$ we obtain such a signed measure on $\mathbb{R}$ also.

Uniqueness of a positive measure, Carleman's theorem

Carleman's theorem (1922–1926): If the real numbers μ_n for $n \geq 0$ satisfy

$$\sum_{i=0}^{k} \sum_{j=0}^{k} c_i c_j \mu_{i+j} \geq 0$$

in all cases, then there is a **unique** non-negative measure ν with $s_n(\nu) = \mu_n$ for all $n \geq 0$ provided that

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/(2n)} = \infty.$$

With $\gamma_{2m} = \inf\{\mu_{2n}^{1/(2n)} : n \geq m\}$, the weaker condition

$$\sum_{n=1}^{\infty} \frac{1}{\gamma_{2n}} = \infty$$

is also sufficient.
The proof is based on considering the analytic function $I(z) = \int_{-\infty}^{\infty} \frac{d\nu(t)}{z-t}$ in half planes.

A uniqueness result for signed measures

Theorem.
Let ν_1 and ν_2 be signed measures on $\mathbb{R}$, neither one of them the zero measure, or concentrated at the origin. Suppose that for all $n \geq 0$ and for $j = 1, 2$, we have $s_n^*(\nu_j) < \infty$, and that for all $n \geq 0$ we have

$$s_n(\nu_1) = s_n(\nu_2).$$

Suppose further that

$$\sum_{n=1}^{\infty} (s_{2n}^*(\nu_1) + s_{2n}^*(\nu_2))^{-1/(2n)} = \infty.$$

Then $\nu_1 = \nu_2$.

Proof of the uniqueness result

Write $\nu_j = \nu_j^+ - \nu_j^-$ for $j = 1, 2$. We have $s_n(\nu_j) = s_n(\nu_j^+) - s_n(\nu_j^-)$, so $s_n(\nu_1) = s_n(\nu_2)$ implies that

$$s_n(\nu_1^+ + \nu_2^-) = s_n(\nu_2^+ + \nu_1^-)$$

for positive measures $\nu_1^+ + \nu_2^-$ and $\nu_2^+ + \nu_1^-$. By Carleman's theorem, we have

$$\nu_1^+ + \nu_2^- = \nu_2^+ + \nu_1^-$$

and hence $\nu_1 = \nu_2$ provided that

$$(*) \qquad \sum_{n=1}^{\infty} (s_{2n}(\nu_1^+ + \nu_2^-))^{-1/(2n)} = \infty.$$

Now

$$s_{2n}(\nu_1^+ + \nu_2^-) \leq s_{2n}^*(\nu_1) + s_{2n}^*(\nu_2),$$

so (*) holds if

$$\sum_{n=1}^{\infty} (s_{2n}^*(\nu_1) + s_{2n}^*(\nu_2))^{-1/(2n)} = \infty.$$

Uniqueness result for real functions

Theorem.
Let $f, g : \mathbb{R} \to \mathbb{R}$ be Lebesgue measureable functions, neither one vanishing almost everywhere. Suppose that for all $n \geq 0$ we have

$$s_n(f) = s_n(g).$$

Suppose that there are positive constants a and C such that for a.e. $x \in \mathbb{R}$ we have

$$\max\{|f(x)|, |g(x)|\} \leq Ce^{-a|x|}.$$

Then

$$f(x) = g(x)$$

for a.e. $x \in \mathbb{R}$. If f and g are continuous on $\mathbb{R}$, then $f(x) = g(x)$ for all x.

We cannot replace the condition $\max\{|f(x)|, |g(x)|\} \leq Ce^{-a|x|}$ by $\max\{|f(x)|, |g(x)|\} \leq Ce^{-a|x|^\beta}$ for any fixed β with $0 < \beta < 1$.

Uniqueness result for complex functions

For complex-valued functions we can define moments in the same way, which are then complex numbers. Applying the above theorem to the real and imaginary parts of functions, we get the following result.

Theorem. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be Lebesgue measureable functions, neither one vanishing a.e. Suppose that for all $n \geq 0$ we have*

$$s_n(f) = s_n(g).$$

Suppose that there are positive constants a and C such that for a.e. $x \in \mathbb{R}$ we have

$$\max\{|f(x)|, |g(x)|\} \leq Ce^{-a|x|}.$$

Then

$$f(x) = g(x)$$

for a.e. $x \in \mathbb{R}$. If f and g are continuous on $\mathbb{R}$, then $f(x) = g(x)$ for all x.

Proof of the uniqueness result for real functions

If $|f(x)| \leq Ce^{-a|x|}$ then

$$s_n^*(f) \leq 2C \int_0^\infty x^n e^{-ax}\, dx = 2C(n!)a^{-n-1}$$

and similarly for g, so that

$$(s_{2n}^*(f) + s_{2n}^*(g))^{-1/(2n)} \geq C_0/n$$

for some fixed $C_0 > 0$. Hence

$$\sum_{n=1}^{\infty} (s_{2n}^*(f) + s_{2n}^*(g))^{-1/(2n)} = \infty.$$

Now the result follows from the one for signed measures.

Example

If $f(x) = e^{-|x|^b}$ where $0 < b < 1$, then with $u = x^b$,

$$s_n^*(f) = 2\int_0^\infty x^n e^{-x^b}\,dx = \frac{2}{b}\int_0^\infty u^{(n-1)/b-1}e^{-u}\,du = \frac{2}{b}\Gamma((n-1)/b)$$

so that $(s_{2n}^*(f))^{-1/(2n)}$ is comparable to $n^{-1/b}$ and $1/b > 1$.

This will not yield a divergent series $\sum_{n=1}^\infty (s_{2n}^*(f) + s_{2n}^*(g))^{-1/(2n)}$.

Thus Carleman's result does not apply. This does not imply that the uniqueness result is not valid, but there are actual counterexamples to show that it is not valid. Hence Carleman's result is almost sharp, and is sharp when it comes to the exponent of $|x|$ in $e^{-|x|^b}$.

Measures all of whose moments vanish

If $s_n(\nu_1) = s_n(\nu_2)$ for all $n \geq 0$, then the signed measure $\nu = \nu_1 - \nu_2$ satisfies $s_n(\nu) = 0$ for all $n \geq 0$. We may have $d\nu = f(x)\,dx$ for some function f. Examples of such functions f:

$f(x) = |x|^{-\log|x|}(1 + A\sin(2\pi \log|x|)) \geq 0$ $(|A| \leq 1)$ (Stieltjes, 1894)

$f(x) = e^{-kx^\alpha}\cos\left(kx^\alpha \tan\frac{\pi\alpha}{2}\right)$
where $\alpha = 2s/(2s+1)$ for a positive integer s and $k = k(\alpha) > 0$. Here $0 < \alpha < 1$ but α could be arbitrarily close to 1 (Shohat–Tamarkin 1943).

Thus the non-negative functions e^{-kx^α} and $e^{-kx^\alpha}\left(1 + A\cos\left(kx^\alpha \tan\frac{\pi\alpha}{2}\right)\right)$, where A is real and $|A| \leq 1$, have the same moments.

Criterion for the positivity of a signed measure

Theorem. *Let* ν *be a signed measure on* $\mathbb{R}$*, not the zero measure. Suppose that for all* $n \geq 0$ *we have* $s_n^*(\nu) < \infty$*, and that*

$$\sum_{n=1}^{\infty} s_{2n}^*(\nu)^{-1/(2n)} = \infty.$$

Suppose further that for all $k \geq 0$ *and for all real numbers* $c_0, c_1, \ldots, c_k$ *we have*

$$\sum_{i=0}^{k} \sum_{j=0}^{k} c_i c_j s_{i+j}(\nu) \geq 0.$$

Then $\nu \geq 0$*.*

Proof of the criterion for the positivity of a signed measure

There is a positive measure ν_1 such that $s_n(\nu_1) = s_n(\nu)$ for all $n \geq 0$. We have

$$s^*_{2n}(\nu_1) = s_{2n}(\nu_1) = s_{2n}(\nu) \leq s^*_{2n}(\nu)$$

so that

$$(s^*_{2n}(\nu) + s^*_{2n}(\nu_1))^{-1/(2n)} \geq 2^{-1/(2n)} s^*_{2n}(\nu)^{-1/(2n)}$$

and hence

$$\sum_{n=1}^{\infty} (s^*_{2n}(\nu) + s^*_{2n}(\nu_1))^{-1/(2n)} = \infty.$$

Hence $\nu = \nu_1$ so that $\nu \geq 0$.

Criterion for the positivity of a function

Theorem. *Let f be a real-valued Lebesgue measurable function defined on $\mathbb{R}$, not equal to zero almost everywhere. Suppose that for some fixed positive numbers C and a, we have*

$$|f(x)| \leq Ce^{-a|x|}$$

for almost every real x. Suppose further that for all $k \geq 0$ and for all real numbers $c_0, c_1, \ldots, c_k$ we have

$$\sum_{i=0}^{k} \sum_{j=0}^{k} c_i c_j s_{i+j}(f) \geq 0.$$

Then $f(x) \geq 0$ for almost all real x. If f is continuous on $\mathbb{R}$, then $f(x) \geq 0$ for all x.

“Small” and “large” measures

In very rough terms, one might understand the above results as follows.

For the moments of a non-zero signed measure to all vanish, the measure must be “large”.

If two different signed measures have the same moments, then their difference must be “large”.

If both signed measures are “small”, then their difference cannot be “large”, so then the signed measures cannot have the same moments.

Criterion for the positivity of a real polynomial

Polynomials do not have finite moments but if P is a real polynomial, we may consider the moments of

$$f(x) = e^{-x^2/2} P(x).$$

Now $P \geq 0$ on $\mathbb{R}$ if, and only if, $f \geq 0$ on $\mathbb{R}$.
We write

$$P(x) = \sum_{j=0}^{N} a_j x^j$$

where all a_j are real. Now

$$s_n(f) = \sum_{j=0}^{N} a_j \int_{-\infty}^{\infty} e^{-x^2/2} x^n x^j \, dx$$

and $\int_{-\infty}^{\infty} e^{-x^2/2} x^n x^j \, dx$ is equal to 0 if $j + n$ is odd, and $\sqrt{2\pi}(n + j - 1)!!$ if $j + n$ is even. ($k!! = k(k-2) \cdots 1$ for odd k)

Criterion for the positivity of a real polynomial, continued

Thus $P \geq 0$ on $\mathbb{R}$ if, and only if, for all $k \geq 0$ and all real $c_0, c_1, \ldots, c_k$ we have

$$\sum_{j=0}^{N} a_j \sum_{p=0}^{k} \sum_{q=0}^{k} c_p c_q \sigma_{p+q+j} \geq 0$$

where $\sigma_j = 0$ if j is odd, and $\sigma_j = (j-1)!!$ if j is even.

Checking by moments whether a polynomial P satisfies $P \geq 0$ on $\mathbb{R}$ leads to conditions that are **linear** in the coefficients of P, just like the condition $P \geq 0$ itself.

Suppose that $P(x)$ is a real polynomial. Suppose that it is not the case that $P \geq 0$ on $\mathbb{R}$. In what manner will the above criteria fail ?

Recall that Δ_k, for $k \geq 0$, is the determinant of the $(k+1) \times (k+1)$ matrix with entries $s_{i+j} = s_{i+j}(f)$, where $f(x) = e^{-x^2/2}P(x)$.

If it is not true that $\Delta_k \geq 0$ for all $k \geq 0$, then there is some $k \geq 0$ such that $\Delta_k < 0$. Thus, if we calculate these determinants, after finitely many steps we find a negative determinant, which shows that the condition "$P \geq 0$ on $\mathbb{R}$" fails.

However, the finite number of steps that we must take may be very large, depending on P, without any specific bound.

Examples of the positivity criterion, remarks on parametrized polynomials

Suppose that one or more coefficients of P depend on a real parameter a.

For any specific value of the parameter (since we have only one polynomial then), if it is not true that "$P \geq 0$ on $\mathbb{R}$", we encounter a negative determinant after finitely many steps. The number of steps may increase as the parameter tends to a value for which $P \geq 0$ on $\mathbb{R}$, or when $|a| \to \infty$.

If the coefficients of P depend continuously on parameters and if for all a in a compact set K in the parameter space the condition "$P \geq 0$ on $\mathbb{R}$" fails, then there is $k \geq 0$ such that for each $a \in K$ there is j with $0 \leq j \leq k$ for which $\Delta_j(a) < 0$.

Suppose that $P(x) = x - a$ where a is real. Then there is no a such that $P \geq 0$ on $\mathbb{R}$. In what manner will the above criteria fail ?

Examples of the positivity criterion for $P(x) = x - a$

Suppose that $P(x) = x - a$ where a is real. Now the determinants Δ_k are polynomials in a. We may divide each s_n by $\sqrt{2\pi}$ before forming Δ_k.

Then $\Delta_0 = -a$, excluding $a > 0$.

We have $\Delta_1 = a^2 - 1$, excluding also $-1 < a \leq 0$. Now only $a \leq -1$ remains.

We have $\Delta_2 = -2a(a^2 - 3) < 0$ for $-\sqrt{3} < a \leq -1$. Now only $a \leq -\sqrt{3}$ remains.

We have $\Delta_3 = 12(3 - 6a^2 + a^4) < 0$ for $-\sqrt{3+\sqrt{6}} < a \leq -\sqrt{3}$. Now only $a \leq -\sqrt{3+\sqrt{6}} \approx -2.33441$ remains.

A bit of thinking shows that no finite number of Δ_k can exclude all real a.

Graphs of determinants for $P(x) = x - a$

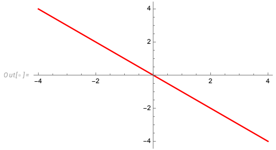

Figure: Δ_0, for $P(x) = x - a$

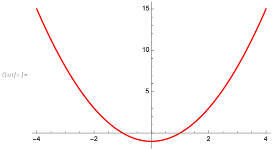

Figure: Δ_1, for $P(x) = x - a$

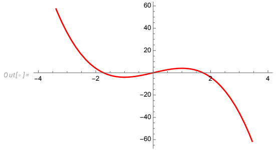

Figure: Δ_2, for $P(x) = x - a$

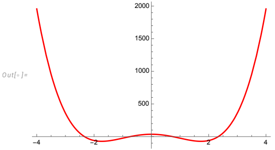

Figure: Δ_3, for $P(x) = x - a$

Using $e^{-|x|}$ instead of $e^{-x^2/2}$

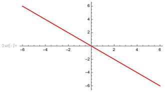

Figure: Δ_0, for $P(x) = x - a$

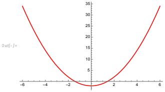

Figure: Δ_1, for $P(x) = x - a$

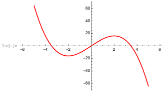

Figure: Δ_2, for $P(x) = x - a$

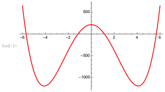

Figure: Δ_3, for $P(x) = x - a$

Now Δ_0 eliminates $a > 0$; Δ_1 (further) eliminates $-\sqrt{2} < a \leq 0$; Δ_2 eliminates $-2\sqrt{3} < a \leq -\sqrt{2}$; Δ_3 eliminates $-A < a \leq -2\sqrt{3}$. Then $a \leq -A$ remains where $A = \sqrt{(6/5)(14 + \sqrt{166})} \approx 5.67987$.

Examples of the positivity criterion for $P(x) = x^2 - a$

Suppose that $P(x) = x^2 - a$ where a is real. Then $P \geq 0$ on $\mathbb{R}$ if, and only if, $a \leq 0$. Thus the above criteria are satisfied for all $a \leq 0$. How do they fail when $a > 0$? Again, the determinants Δ_k are polynomials in a.

If $a > 0$ and a is close to 0, then the failure is small, which is captured by how many Δ_k must be calculated to find that failure.

When $a = 0$, every $\Delta_k > 0$ (since the positive measure $f(x)\,dx$ is not supported by a finite set of x), so that there is $\varepsilon_k > 0$ such that $\Delta_k(a) > 0$ for all a with $0 < a < \varepsilon_k$. Thus no finite number of Δ_k can exclude all $a > 0$.

Graphs of determinants for $P(x) = x^2 - a$

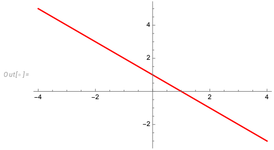

Figure: Δ_0, for $P(x) = x^2 - a$

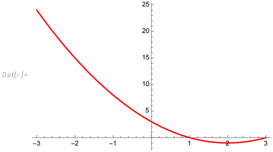

Figure: Δ_1, for $P(x) = x^2 - a$

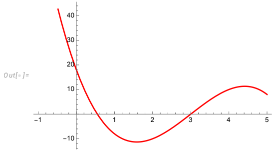

Figure: Δ_2, for $P(x) = x^2 - a$

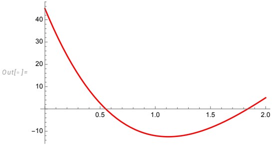

Figure: Δ_3, for $P(x) = x^2 - a$

Determinants for $P(x) = x^2 - a$

We have $P(x) = x^2 - a \geq 0$ on $\mathbb{R}$ if, and only if, $a \leq 0$. Thus each $a > 0$ needs to be eliminated after some finite number of steps.

We have $\Delta_0 = 1 - a$, which eliminates $a > 1$.

We have $\Delta_1 = 3 - 4a + a^2$, which does not eliminate any further a.

We have $\Delta_2 = -2(-9 + 21a - 9a^2 + a^3)$, which further eliminates $3 - \sqrt{6} < a \leq 1$. We have $3 - \sqrt{6} \approx 0.55051$.

We have $\Delta_3 = 12(45 - 120a + 78a^2 - 16a^3 + a^4)$, which does not eliminate any further a.

We have $\Delta_4 = -288(-225 + 825a - 690a^2 + 210a^3 - 25a^4 + a^5)$, which further eliminates $0.380327 < a \leq 3 - \sqrt{6}$.

Using $e^{-x^2/2}$ for $P(x) = x^3 - a$

$\Delta_0 = -a$ eliminates $a > 0$; $\Delta_1 = a^2 - 9$ also $-3 < a \leq 0$;
$\Delta_2 = -2a(-81 + a^2)$ also $-9 < a \leq -3$.

When we get to $\Delta_3 = 12(675 - 330a^2 + a^4)$, only

$$a \leq -\sqrt{15(11 + \sqrt{118})} \approx -18.1092$$

remains.

When we get to $\Delta_4 = -288a(20475 - 930a^2 + a^4)$, only

$$a \leq -\sqrt{15(31 + \sqrt{870})} \approx -30.1237$$

remains.

Elimination happens “faster” than for $P(x) = x - a$.

Positive polynomials in several variables

The criteria can be used for real polynomials in any finite number of variables. To see how, consider two variables. Then

$$P(x,y) = \sum_{i=0}^{M}\sum_{j=0}^{N} a_{i,j}x^i y^j.$$

First fix one of x and y, say, $x = x_0$. Set

$$f(y) = e^{-y^2/2}\sum_{i=0}^{M}\sum_{j=0}^{N} a_{i,j}x_0^i y^j$$

and apply the previous results to this function of y. We find that $f(y) \geq 0$ for all real y if, and only if,

$$\sum_{i=0}^{M}\sum_{j=0}^{N} a_{i,j}x_0^i \sum_{p=0}^{k}\sum_{q=0}^{k} c_p c_q \sigma_{p+q+j} \geq 0$$

where $\sigma_j = 0$ if j is odd, and $\sigma_j = (j-1)!!$ if j is even.

Positive polynomials in several variables, continued

This needs to be true for all real x_0, so replace x_0 by x, and note that the function

$$\sum_{i=0}^{M}\sum_{j=0}^{N} a_{i,j}x^i \sum_{p=0}^{k}\sum_{q=0}^{k} c_p c_q \sigma_{p+q+j}$$

is a polynomial in x. Now

$$g(x) = e^{-x^2/2}\sum_{i=0}^{M}\sum_{j=0}^{N} a_{i,j}x^i \sum_{p=0}^{k}\sum_{q=0}^{k} c_p c_q \sigma_{p+q+j} \geq 0$$

for all real x if, and only if, for all real d_r, we have

$$\sum_{i=0}^{M}\sum_{j=0}^{N} a_{i,j} \sum_{r=0}^{\ell}\sum_{s=0}^{\ell} d_r d_s \sum_{p=0}^{k}\sum_{q=0}^{k} c_p c_q \sigma_{p+q+j}\sigma_{r+s+i} \geq 0.$$

Positive polynomials in several variables, continued

If we start with a polynomial $P = P(x_1, \ldots, x_d)$, then, one by one, the real variables x_i, which can appear in the condition $P \geq 0$ raised to an arbitrarily large power (depending on P), are replaced by new variables such as $c_1, \ldots, c_n$, each of which appears in the resulting conditions only **quadratically**.

The number of conditions becomes infinite.

Sums of squares of polynomials

In one variable, a polynomial is positive if, and only if, it can be expressed as the sum of squares of two polynomials (Pólya).

In at least two variables, by the solution to Hilbert's 17th problem (Artin, 1927), a positive polynomial (i.e., $P(x_1, \ldots, x_d) \geq 0$ for all real $x_1, \ldots, x_d$) can be expressed as the sum of squares of finitely many rational functions, but not always (Hilbert, 1893, abstract proof) as the sum of squares of finitely many polynomials.

Some counterexamples:
$1 - 3X^2Y^2 + X^2Y^4 + X^4Y^2$ (Motzkin, 1967),
$1 - X^2Y^2 + X^2Y^4 + X^4Y^2$ (Berg, Christensen, Jensen, 1979).

If $P(x_1, \ldots, x_d)$ is a positive polynomial, then for some $N \geq 0$ the function

$$(1 + x_1^2 + x_2^2 + \cdots + x_d^2)^N P(x_1, \ldots, x_d)$$

can be expressed as the sum of squares of finitely many polynomials (Pólya, 1928; Reznick, 1996).

Constructing examples of positive polynomials

Consider a polynomial that is certainly ≥ 0 on $\mathbb{R}$:

$$P(x) = \left(\prod_{i=1}^{M} (x - x_i)^2 \right) \left(\prod_{j=1}^{N} \left((x - y_j)^2 + b_j^2 \right) \right),$$

where the x_i, y_j, b_j are real parameters. Form the moments

$$s_k = s_k(e^{-x^2/2} P(x))$$

and consider the determinants

$$\Delta_k = \det |s_{i+j}|_{i,j=0}^{k} > 0.$$

The Δ_k are polynomials in the variables x_i, y_j, b_j that are > 0 for all real values of these variables, hence **positive polynomials**.

Constructing examples of positive polynomials, continued

Example: $P(x) = (x-a)^2((x-b)^2 + c^2)$.

Then

$$\Delta_0 = 3 + 4ab + b^2 + c^2 + a^2(1 + b^2 + c^2),$$

$$\begin{aligned}\Delta_1 = {} & 45 - 12a^2 + 3a^4 + 24ab - 12b^2 + 24a^2b^2 + 8a^3b^3 \\ & + 3b^4 + a^4b^4 + 24c^2 + 4a^4c^2 + 8a^3bc^2 + 6b^2c^2 \\ & + 2a^4b^2c^2 + 3c^4 + a^4c^4.\end{aligned}$$

Constructing examples of positive polynomials, continued

Example: $P(x) = (x-a)^2(x-b)^2$.

Then

$$\Delta_0 = 3 + 4ab + b^2 + a^2(1+b^2),$$

$$\Delta_1 = 45 - 12a^2 + 3a^4 + 24ab - 12b^2$$
$$+24a^2b^2 + 8a^3b^3 + 3b^4 + a^4b^4.$$

Constructing examples of positive polynomials, continued

Example: $P(x) = (x-a)^2(x-b)^2(x-1)^2$.

Then

$$\Delta_0 = 2(9 + 2b(3+b) + 2a(1+b)(3+b) + a^2(2 + b(2+b))),$$

$$\begin{aligned}\Delta_1 = {} & 1260 + 360a - 132a^2 + 24a^3 + 36a^4 + 360b \\ & + 624ab + 336a^2b + 96a^3b + 24a^4b - 132b^2 + 336ab^2 \\ & + 456a^2b^2 + 144a^3b^2 + 12a^4b^2 + 24b^3 + 96ab^3 \\ & + 144a^2b^3 + 80a^3b^3 + 8a^4b^3 + 36b^4 + 24ab^4 + 12a^2b^4 \\ & + 8a^3b^4 + 4a^4b^4.\end{aligned}$$

These look like sums of squares of polynomials.

Polynomials with only real zeros

Pólya proved that if P is a real polynomial of degree $n \geq 2$, then P has only real zeros if, and only if, each of the following $n-1$ polynomials is ≥ 0 on $\mathbb{R}$:

$$(n-j)(P^{(j)})^2 - (n-j+1)P^{(j-1)}P^{(j+1)}$$

for $1 \leq j \leq n-1$. For example, if $j=1$, the inequality reads

$$(n-1)(P')^2 - nPP'' \geq 0,$$

and if $j = n-1$, the inequality reads

$$(P^{(n-1)})^2 - 2P^{(n-2)}P^{(n)} \geq 0.$$

A strict inequality holds in all cases if the zeros of P are also all simple. To test this, one can consider the moments of the functions

$$f_j(x) = e^{-x^2/2}((n-j)(P^{(j)}(x))^2 - (n-j+1)P^{(j-1)}(x)P^{(j+1)}(x))$$

for $1 \leq j \leq n-1$.

Polynomials with only real zeros, continued

The consideration of the moments of the functions

$$f_j(x) = e^{-x^2/2}((n-j)(P^{(j)}(x))^2 - (n-j+1)P^{(j-1)}(x)P^{(j+1)}(x))$$

for $1 \leq j \leq n-1$ leads to conditions that are **quadratic** in the coefficients of P.

Checking by moments whether a polynomial is ≥ 0 on $\mathbb{R}$ leads to conditions that are **linear** in the coefficients of P.

Real entire functions with only real zeros

Let

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

be a real entire function of genus 0 or 1, not of the form $e^{az}P(z)$ where P is a polynomial.

Pólya proved that all the zeros of F are real if, and only if, for all $n \geq 1$ and all real x we have

$$(F_n(x))^2 - F_{n-1}(x)F_{n+1}(x) > 0$$

where the polynomials F_n are defined by

$$F_n(z) = a_0 z^n + \binom{n}{1} a_1 z^{n-1} + \binom{n}{2} a_2 z^{n-2} + \cdots + a_n.$$

These conditions can also be tested by using the moments of $e^{-x^2/2}((F_n(x))^2 - F_{n-1}(x)F_{n+1}(x))$.

 

THANK YOU!