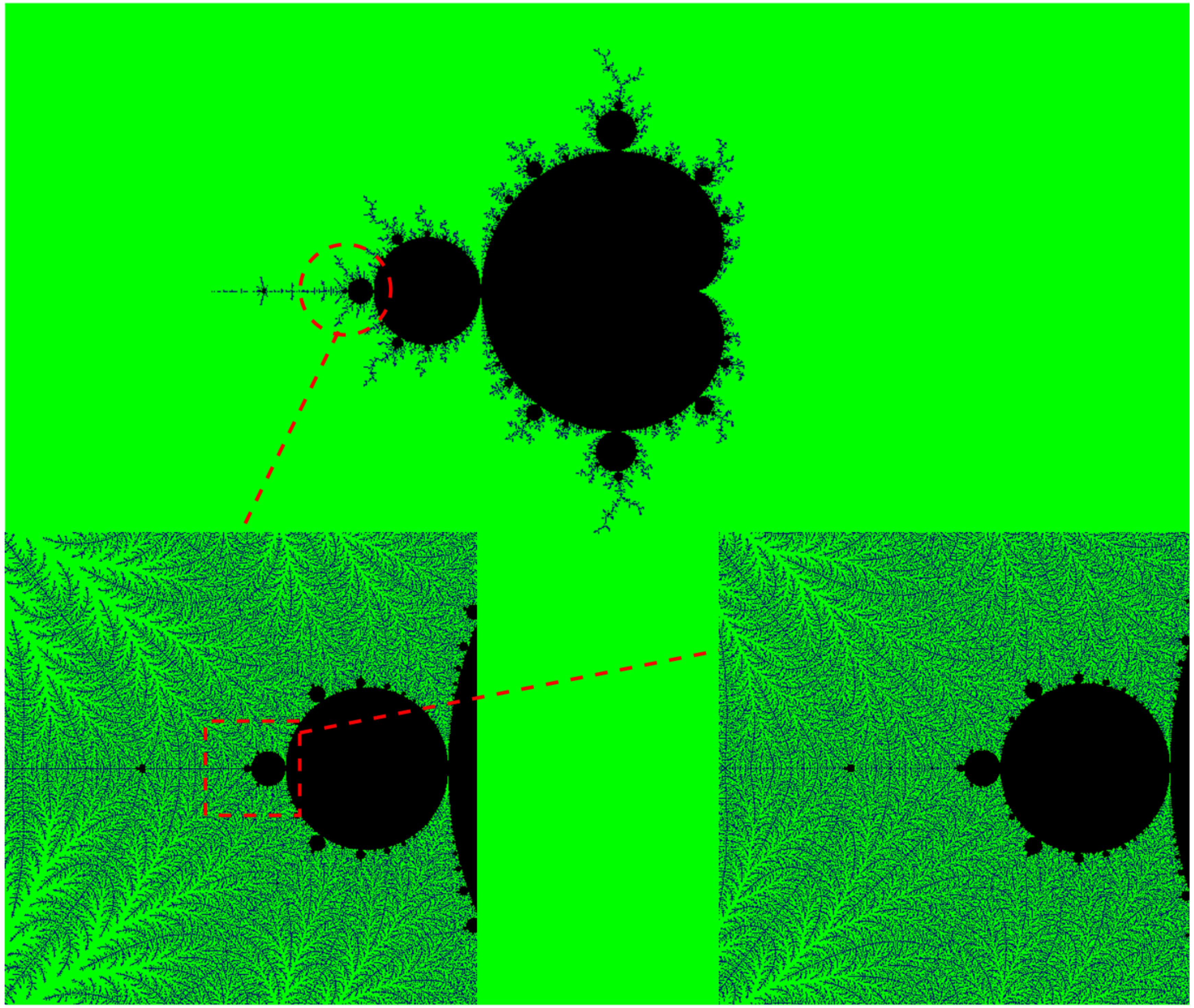
MLC at Feigenbaum points

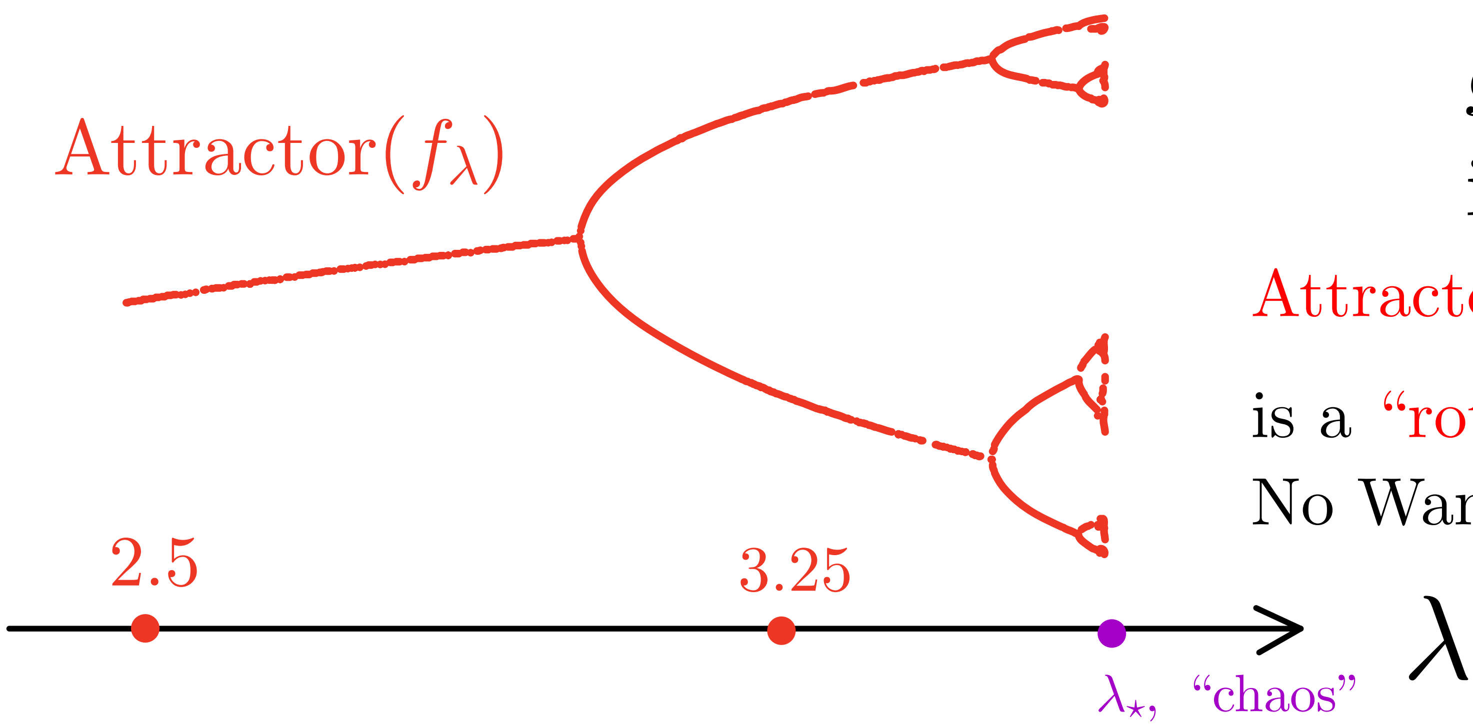
University of Wisconsin-Milwaukee, AMS Sectional Meeting,

Apr 21, 2024



Dzmitry Dudko, Stony Brook University

Logistic maps: $f_\lambda(x) = \lambda x(1-x)$

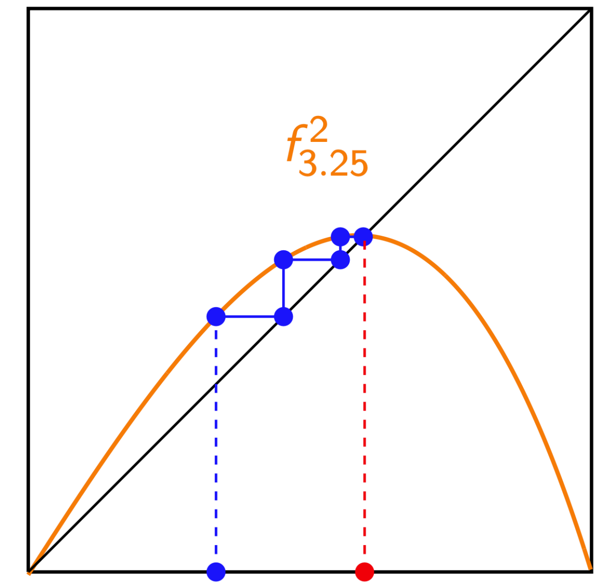


$g_\mu(x) = \mu \sin x$

is a similar family

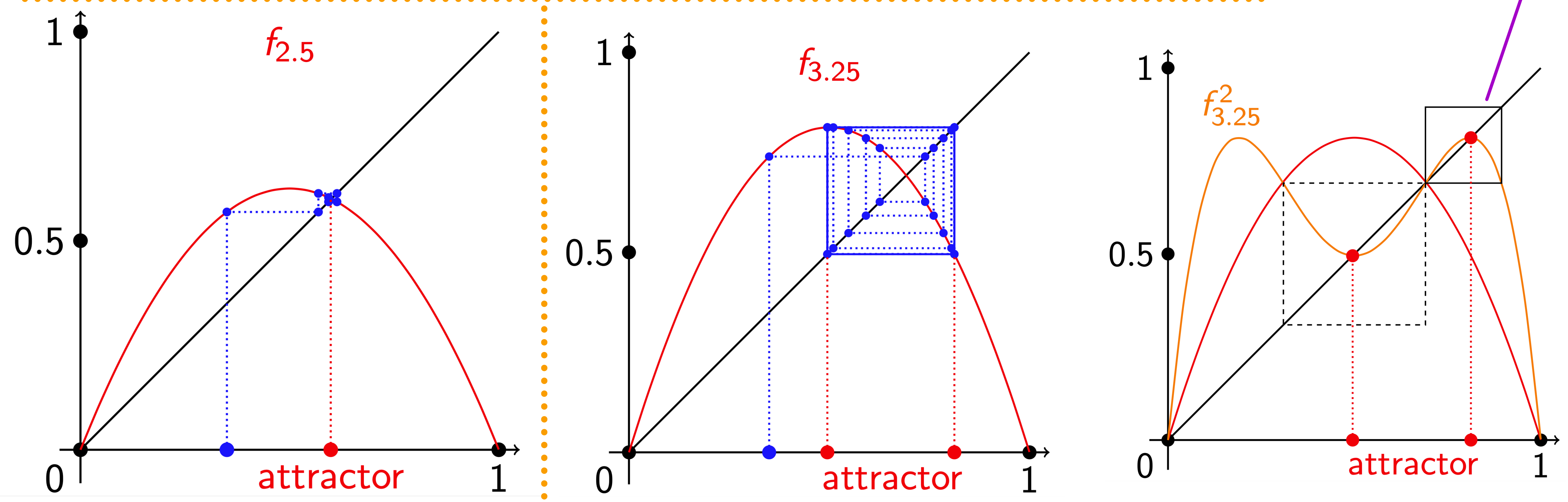
$\text{Attractor}(f_{\lambda_\star}) \simeq \varprojlim \, (\mathbb{Z}/2^n, +)$

is a "rotating" Cantor set (by No Wandering Intervals Thm)



Renormalization

$f_{3.25} \mapsto f_{3.25}^2$



The chaos (sensitive dependence on initial conditions) was discovered by Poincaré in the 1880s (three body problem) but was not in agenda for 70 years.

Logistic maps: $$f_\lambda(x) = \lambda x(1-x)$$

the first family where chaos was studied and fully understood

1940s: Ulam–Neumann λ=4,

1960s: Lorenz, Myrberg, Sharkovsky,

1970s, Milnor–Thurston: Full Combinatorial Theory

Misiurewicz, Guckenheimer

No Wandering Intervals Theorem

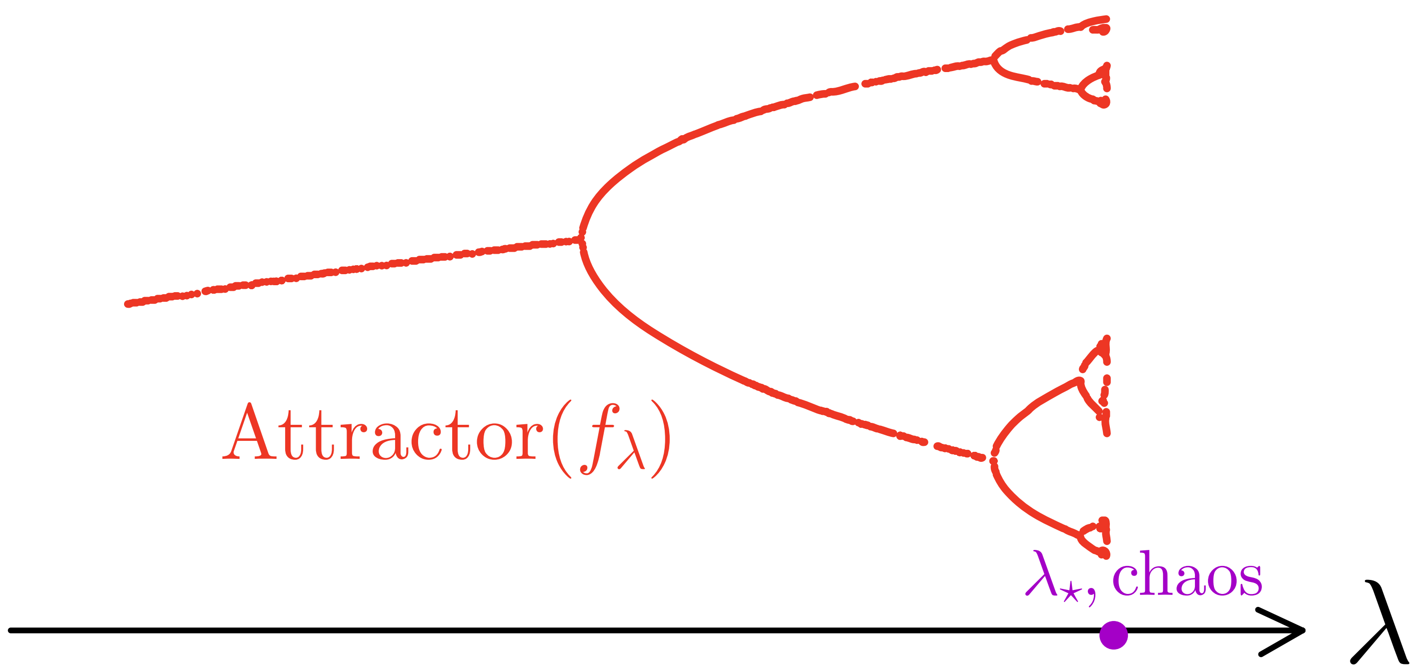
popularized in the mid–70s by R. May as a discrete model of the logistic equation

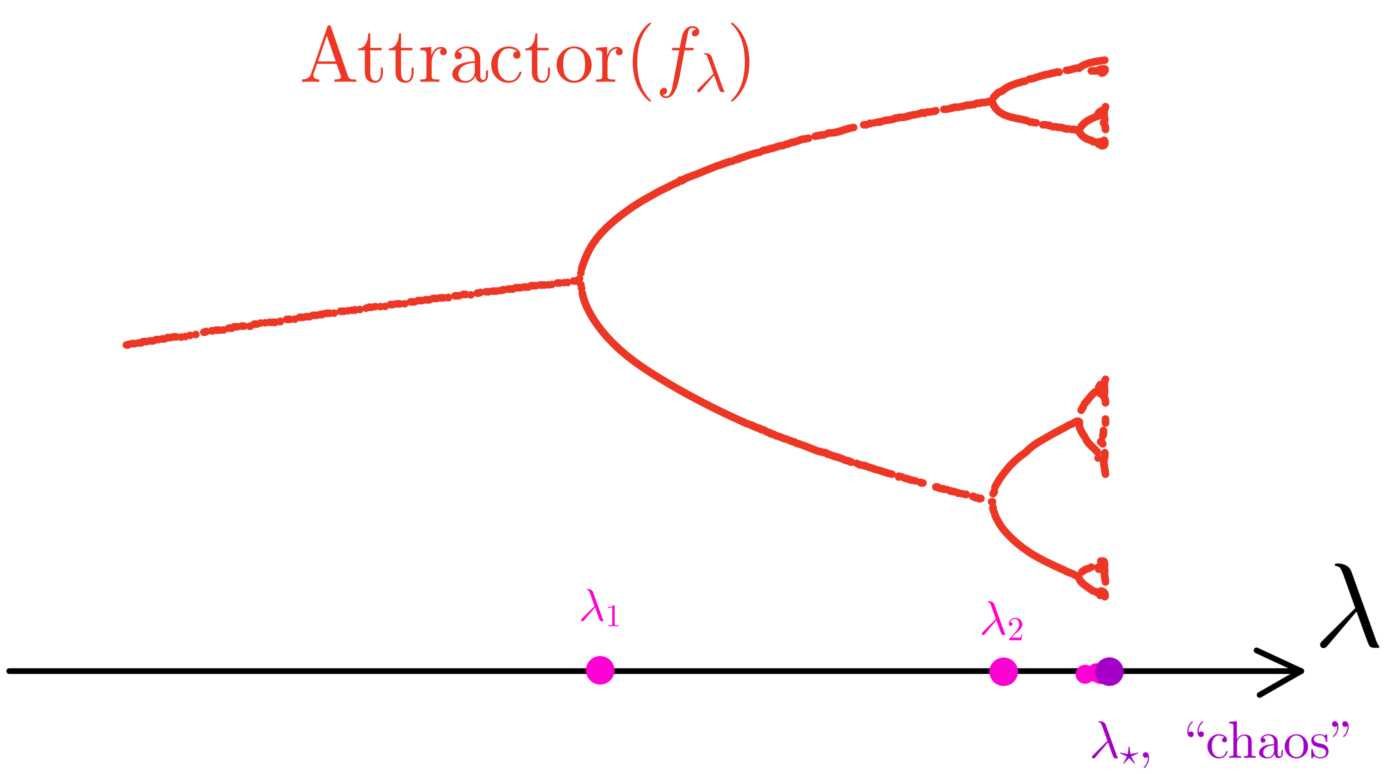
$$f' = f(1-f)$$

(has no chaos)

population growth,

Pierre F. Verhulst, 1840s





the mid-late 1970s, Discoveries:

Feigenbaum, parameter universality:

$$\exists \lim_{n\to\infty} \frac{\lambda_{n-1}-\lambda_{n-2}}{\lambda_n-\lambda_{n-1}} = 4.6692\ldots$$

it is the same number for the

$g_\mu(x) = \mu \sin x$ family

independently and simultaneously,
Coullet-Tresser, dynamical universality:

$\mathrm{Attractor}(\mathrm{f}_{\lambda_\star}) \simeq \varprojlim (\mathbb{Z}/2^n, +)$ is a "rotating" Cantor set or a dyadic adding maching (by the No Wandering Intervals Thm)

HD[Attractor($f_{\lambda_\star}$)] is universal, f. e., HD[Attr($f_{\lambda_\star}$)] = HD[Attr($g_{\mu_\star}$)]

Lanford, early 1980s, computer-assisted proof

Rigidity, exm : a qc conjugacy h between $f_{\lambda_\star}$ and $g_{\mu_\star}$ is $C^{1+\alpha}$ conformal on the Attractor :

$$h(x+\Delta x) = h(x) + h'(x)\Delta x = O(|\Delta x|^{1+\alpha})$$

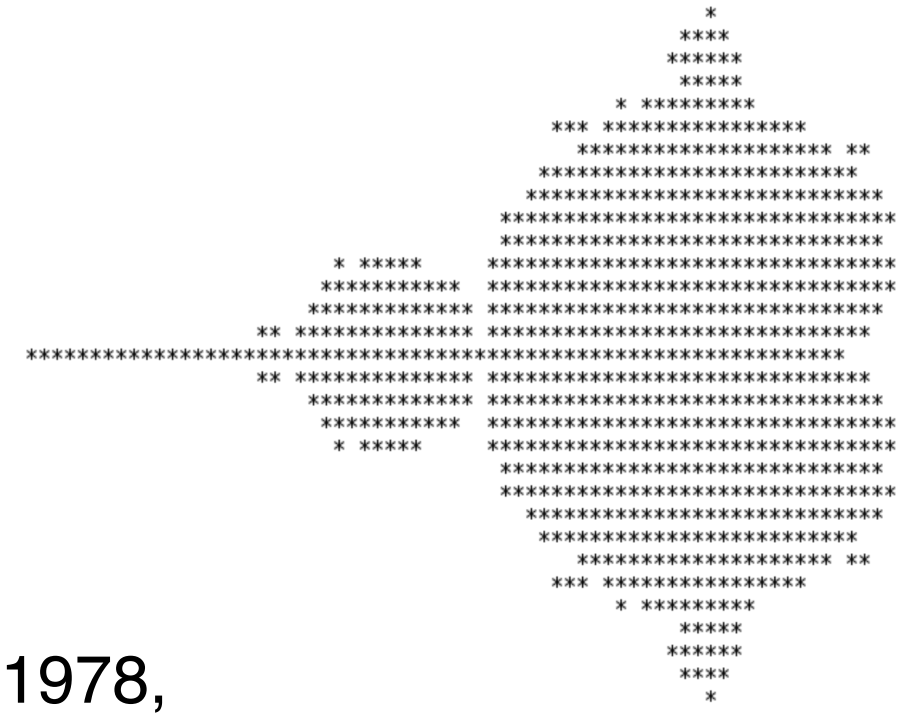
it is similar to $C^{1+\alpha}$-rigidity of circle diffeomorphisms with diophantine rotation numbers, Herman, late 1970s

Sullivan, late-80s + early-90s, a priori bounds, and Teichmüller Theory of the Renormalization (Real Dynamics but with Complex Methods relying on the Douady-Hubbard quadratic-like theory)

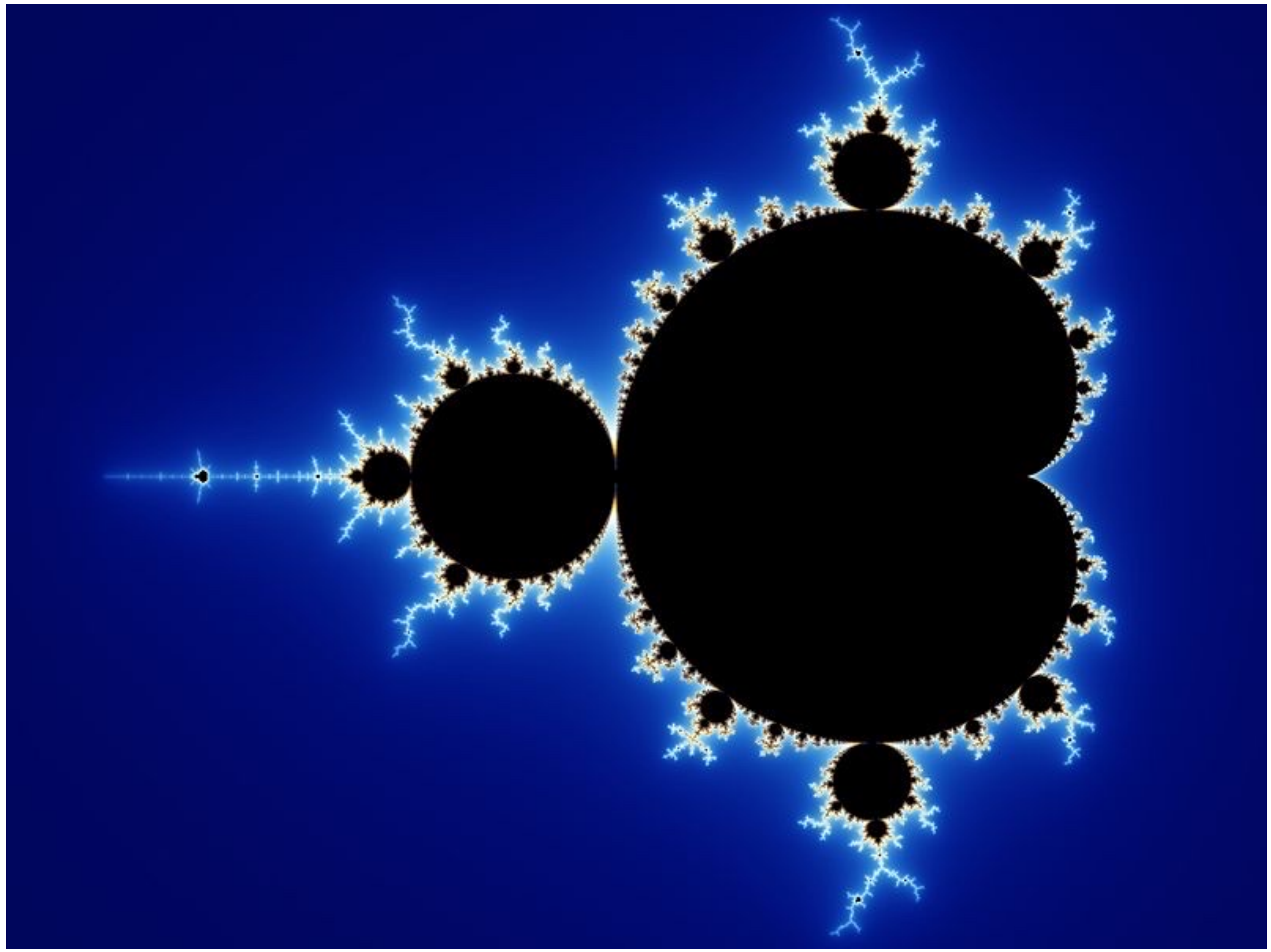
The Mandelbrot set

$$\mathcal{M} = \{c \mid \text{Julia set } J_c \text{ of } f_c(z) = z^2 + c \text{ is connected}\}$$

$$c \notin \mathcal{M}, \quad \text{iff} \quad J_c \text{ is a Cantor set}$$



1978,
by Brooks and Matelski



Douady, Hubbard, 80s:

$\mathcal{M}$ is connected
has ∞-many copies of itself
has rich but understandable combinatorics, . . .
they put forward the MLC conjecture

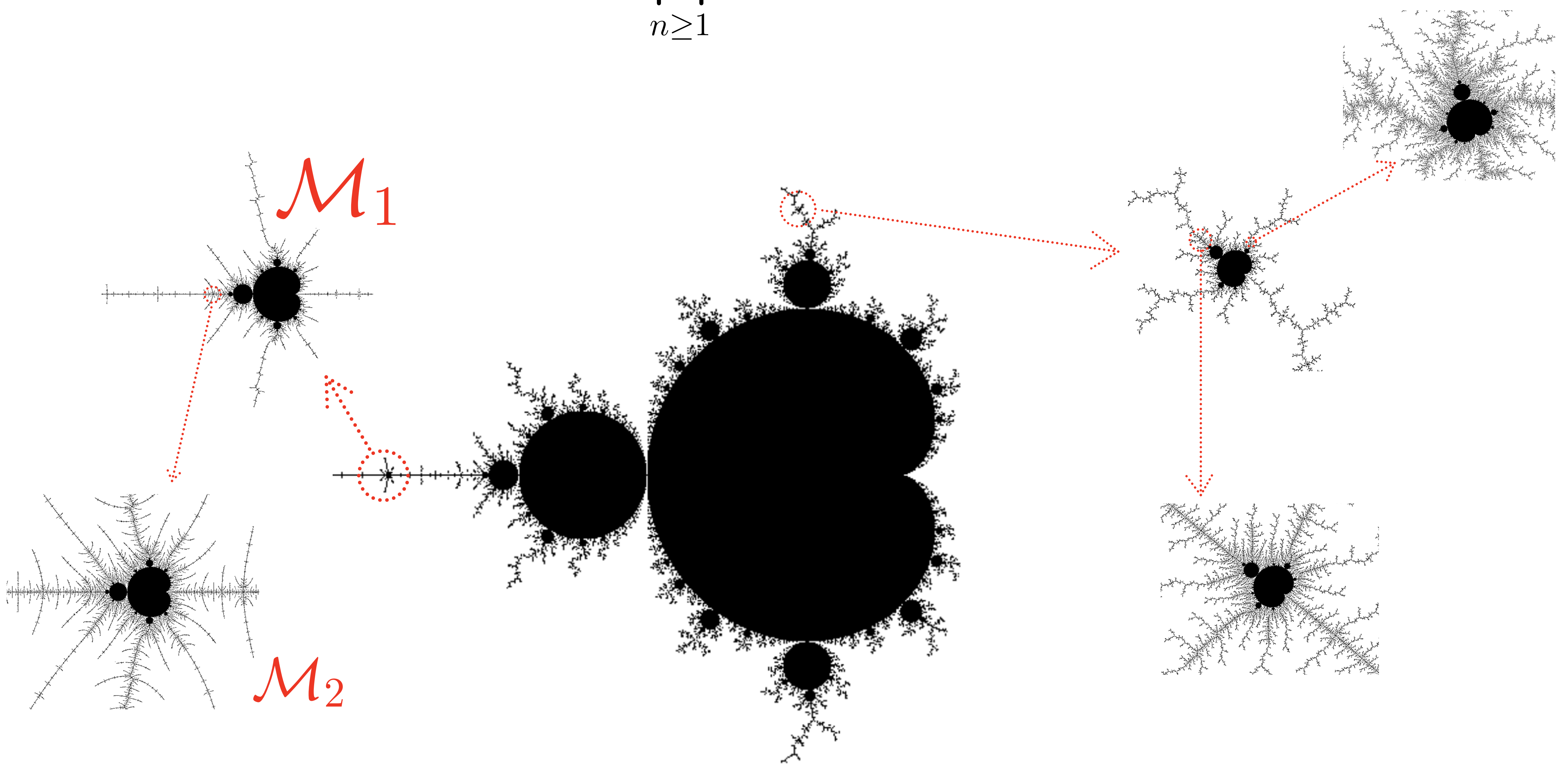
The MLC-conjecture: the Mandelbrot set is locally connected

Yoccoz, early 90s MLC holds at non-infinitely renormalizable parameters

i.e, MLC is equivalent to shrinking of small

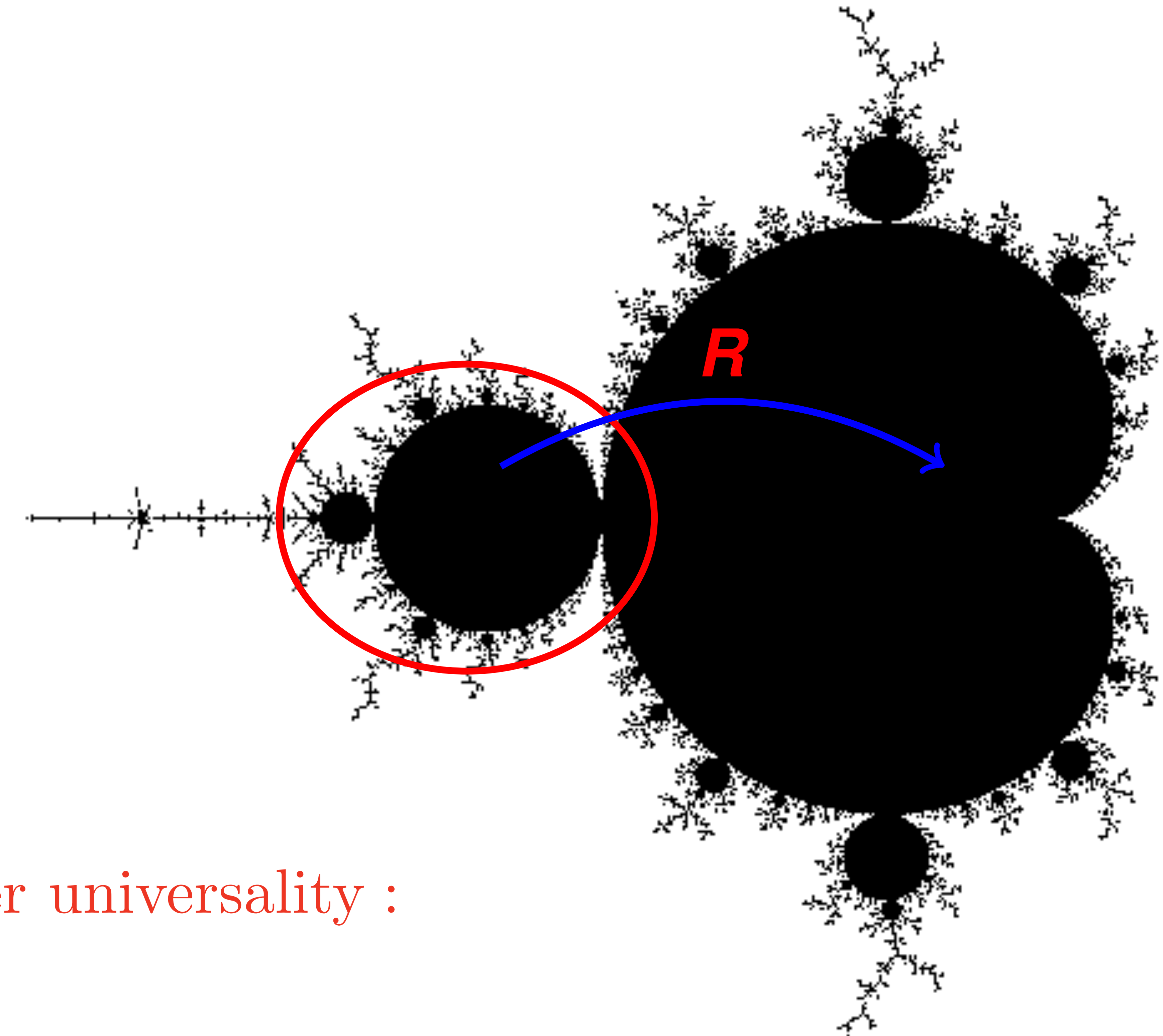
if $\mathcal{M} \supsetneq \mathcal{M}_1 \supsetneq \mathcal{M}_2 \supsetneq \mathcal{M}_3 \supsetneq \dots$

then $\bigcap_{n\geq 1} \mathcal{M}_n = \{c_*\}$ is a singleton



Douady, Hubbard : $\mathcal{M}$ has infinitely many copies of itself canonically homeomorphic to itself

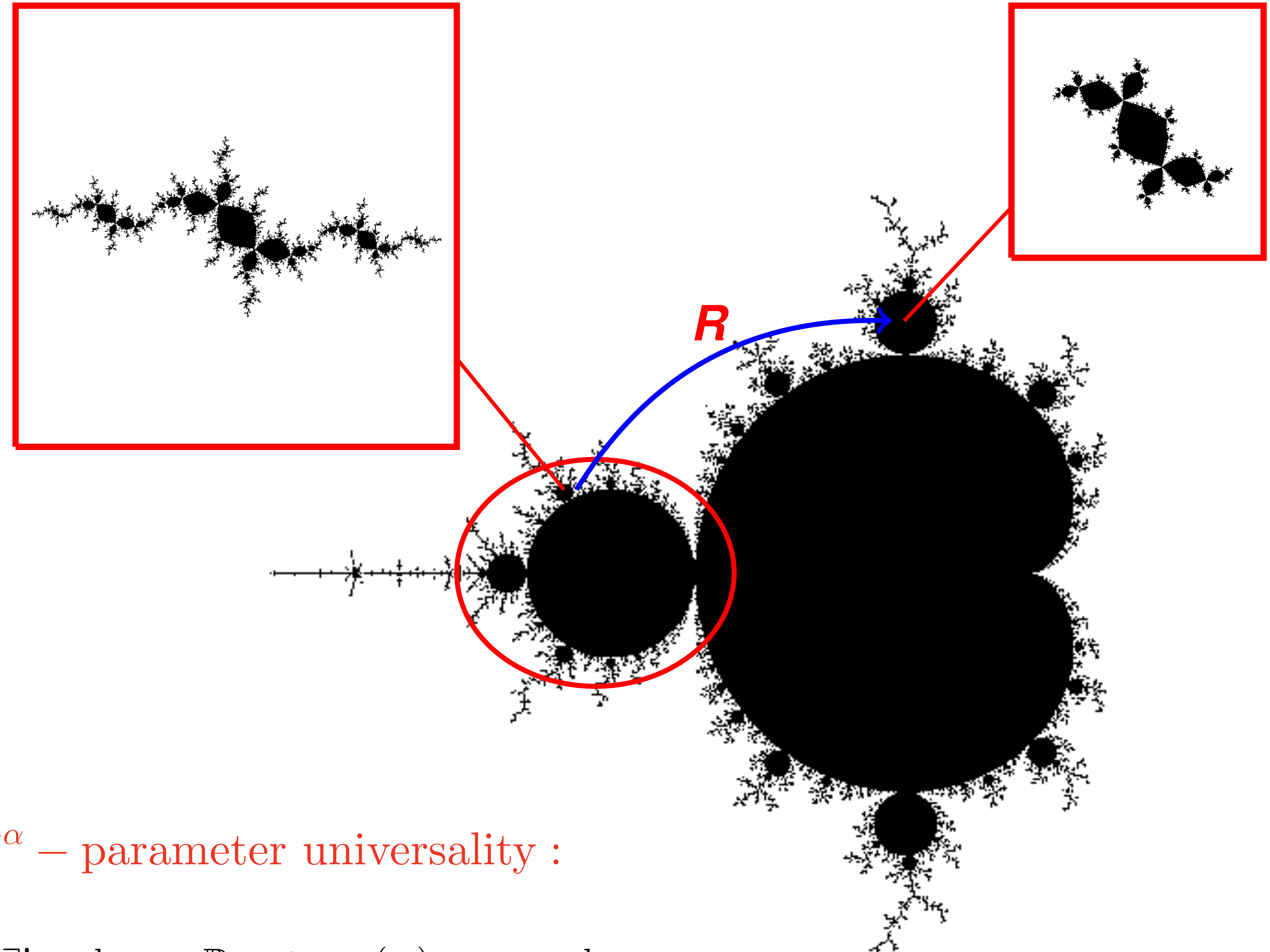
Canonical homeomorphism:



$C^{1+\alpha}$ – parameter universality :

$\exists!$ real $c_\star \in \mathbb{R}$ s.t. $\mathbf{R}(c_\star) = c_\star$ and

$\mathbf{R}(c_\star + w) = \mathbf{R}(c_\star) + \mathbf{R}'(c_\star)\, w + o\left(|w|^{1+\alpha}\right)$



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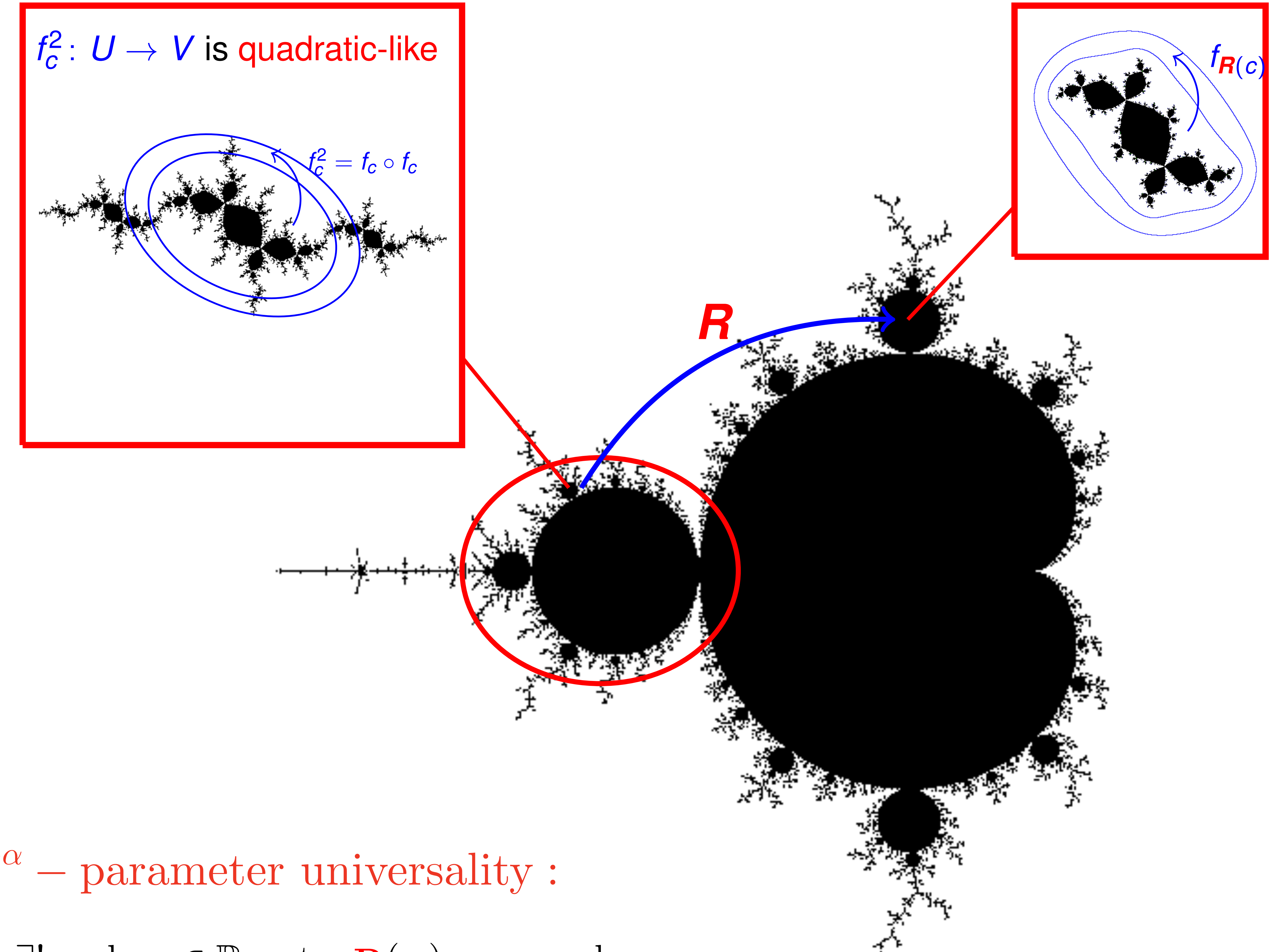

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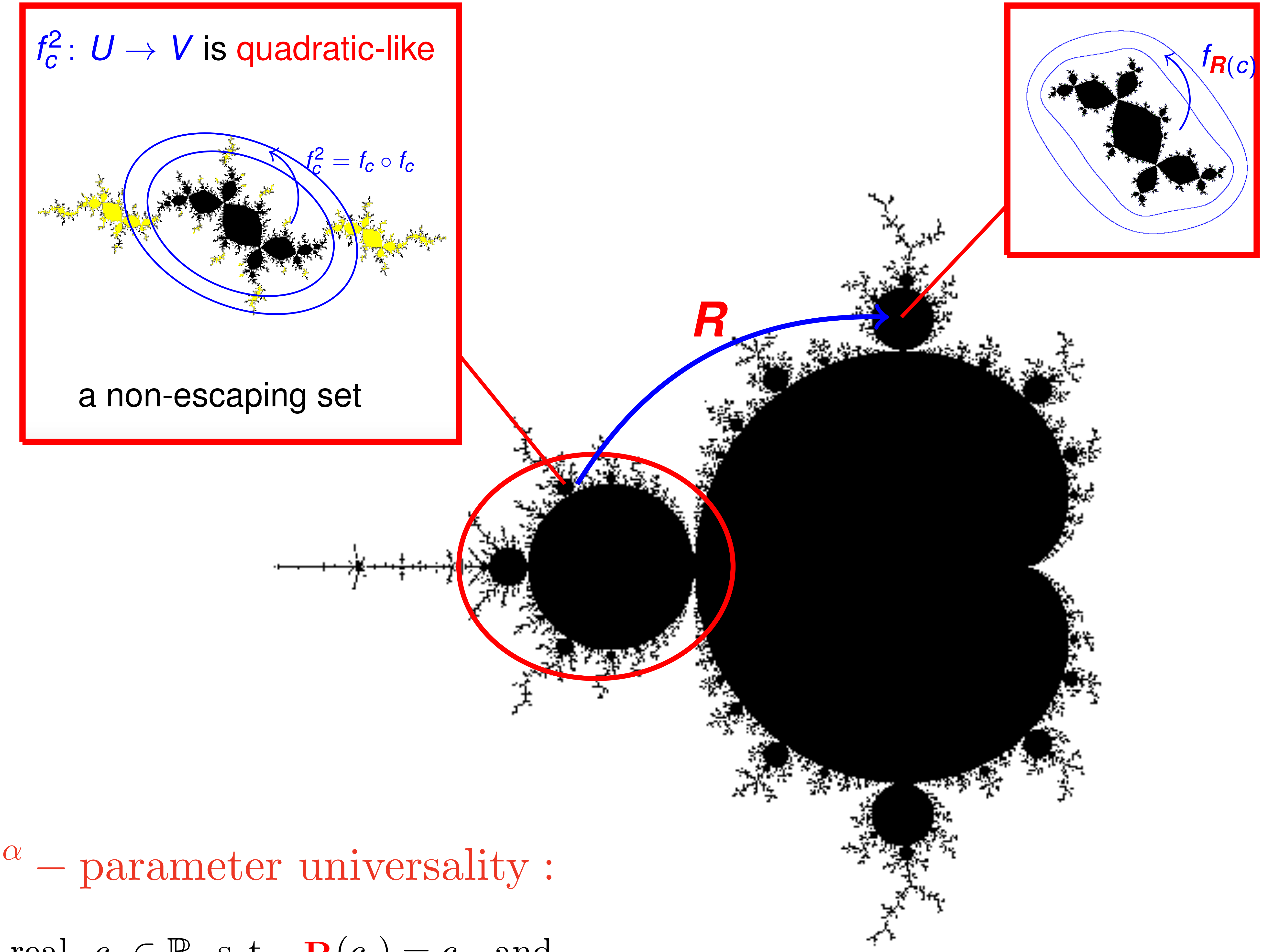

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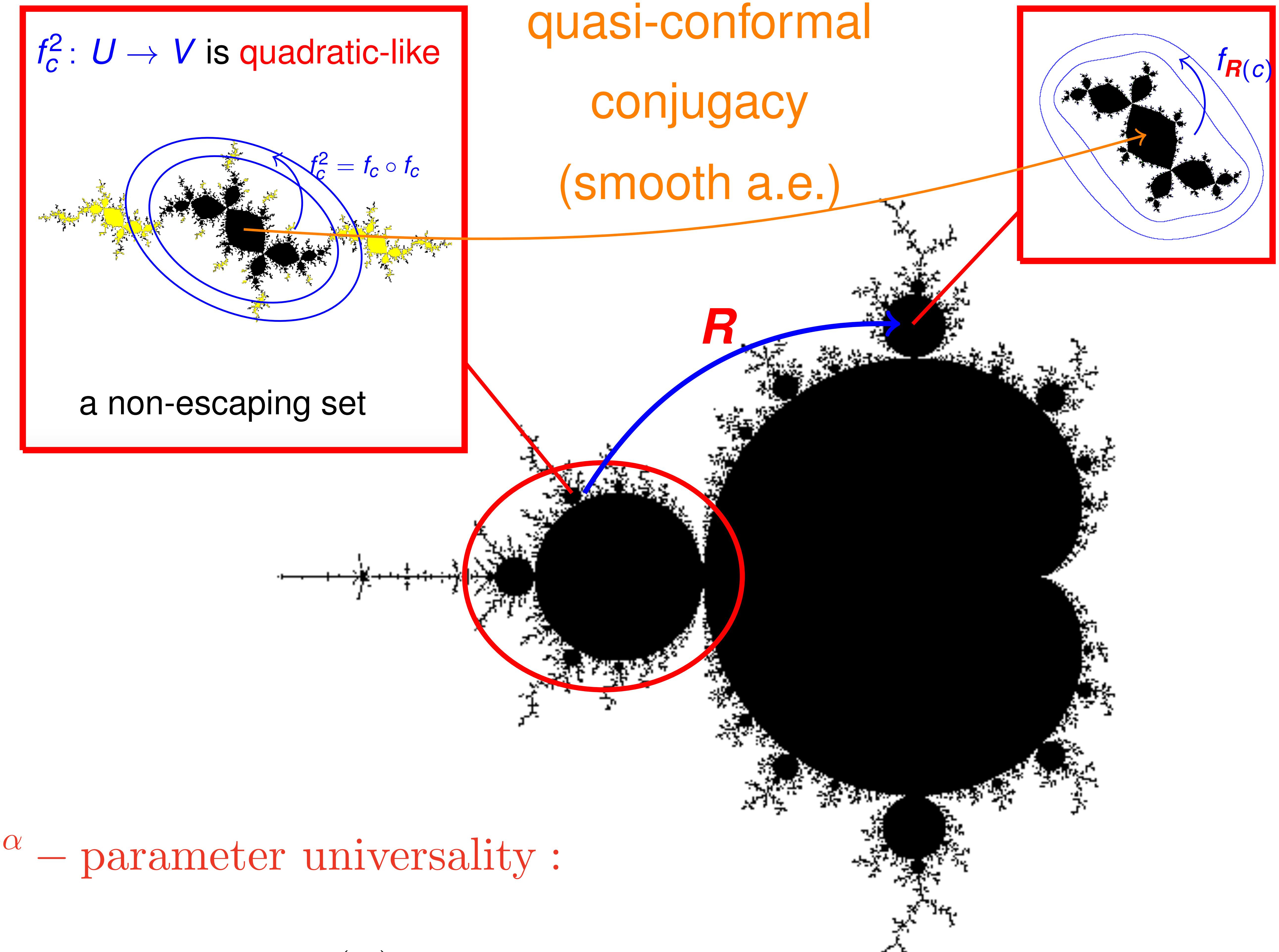

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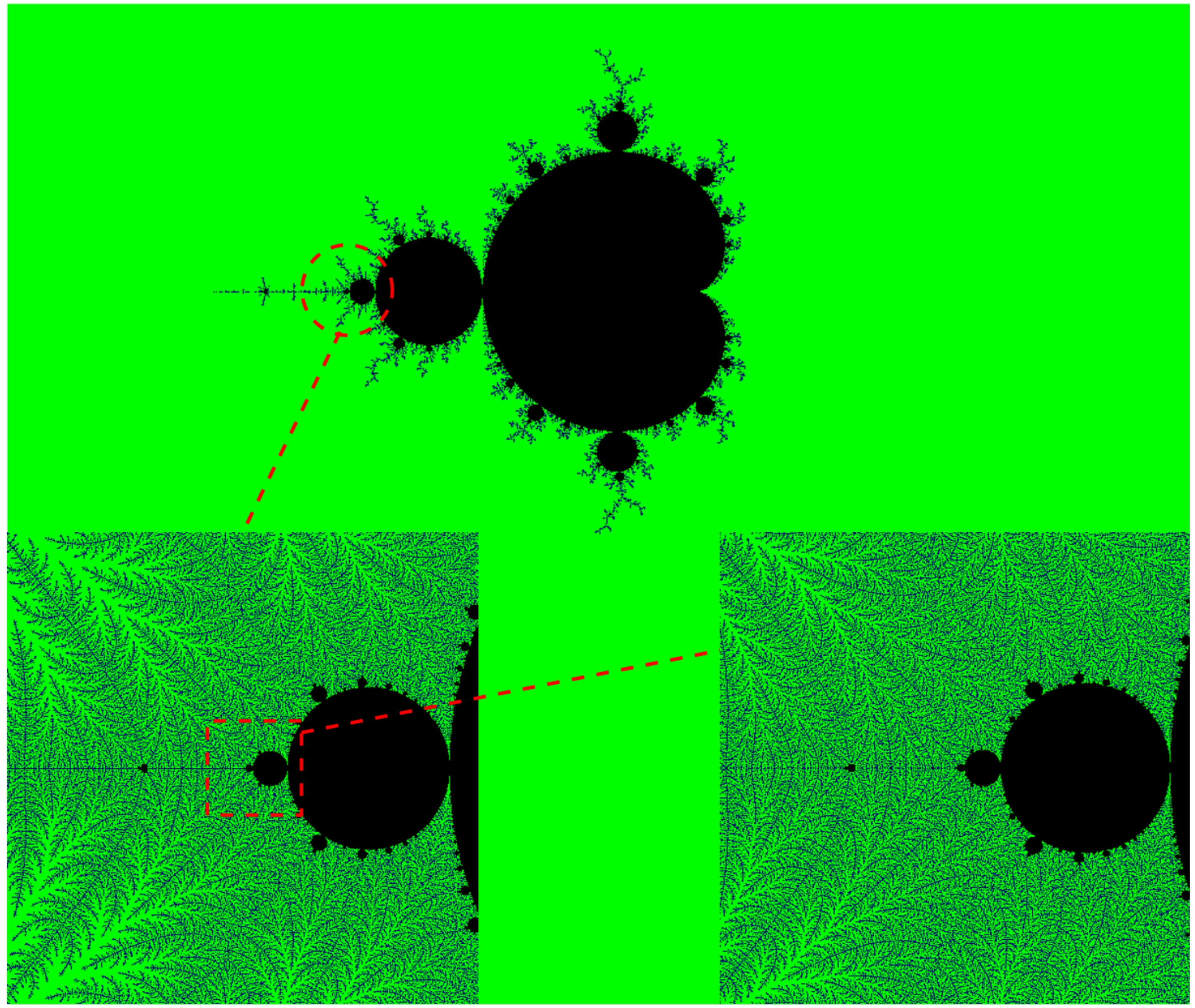
Sullivan, 80s -- early 90s
a priori bounds, renorm fixed point

McMullen, mid 90s
dynamical universality

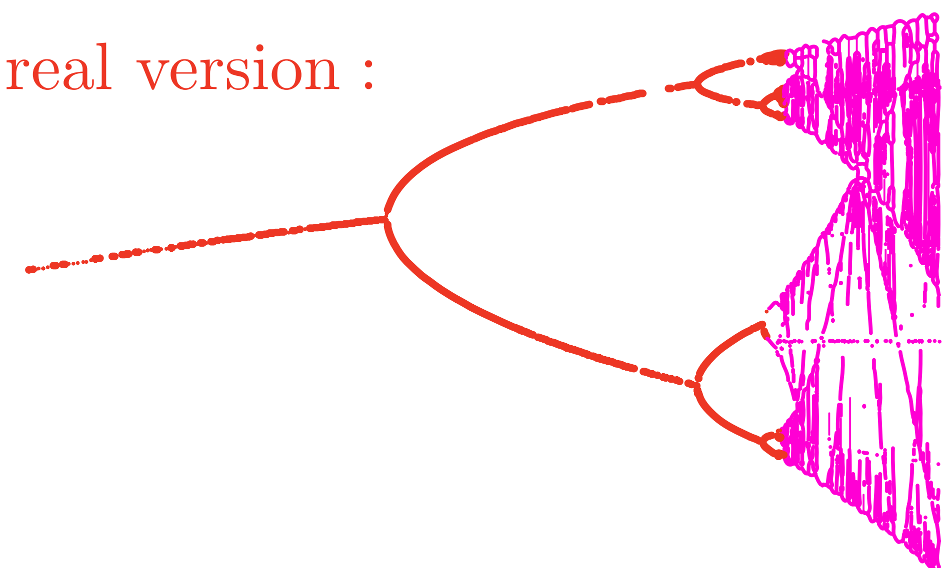
Lyubich, late 90s
parameter universality

same statements hold for any other small copy on the real line

MLC remained open at that time for the period-doubling Feigenbaum, parameter
(and for similar parameters)



real version :



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A lot has been achieved in the 1990s, f.e.:

1) a good understanding of quadratic-like renormalization and respective universalities
Sullivan, McMullen, Lyubich

2) basics of the Siegel and Parabolic renormalizations

applications of Parabolic renorm: Shishikura : $\dim_{\mathrm{HD}} \partial\mathcal{M} = 2$ conjecturally, $\mathrm{Area}(\partial\mathcal{M}) = 0$

3) full understanding of the real polnomials $z^2 + c, \ c \in \mathbb{R}$ – the logistic family

density of hyperbolicity for real pol-ls
Lyubich; Graczyk and Swiatek:

a.e. real pol-l is either regular or stochastic
Lyubich (probability approach inspired by Kolmogorov)

+ various generalizations to higher degree real polynomials...

Major remaining challenges towards the MLC left from the 1990s:

a priori bounds

it became apparent that a priori bounds (precompactness of the first return maps) is a main step for the MLC

non-JLC parameters

Local connectivity of the Julia Set (JLC) fails for some parameters; understanding non-JLC phenomenon is essential for the MLC late 2000s, two new theories emerged based on new principals:

Kahn-Lyubich,
Near-Degenerate regime
a machinery to produce a priori bounds
could not handle non-JLC parameters
at the time
sort of a ``topological dynamics'' but with
non-crossing constrains (more later)

Inou-Shishikura
near-Parabolic Renormalization
(responsible for non-JLC)
perturbative theory allowing to deals with
most delicate non-JLC parameters
instead of the original map, one iterates
the renormalization change of variables

both theories had multiple applications

but were in many ways ``incompatible''
(different languages, different tools,
different objects/regimes...)

It took some time to find a way to combine the ideas of the theories:

DD, Lyubich 2022: Uniform a priori bounds for neutral renormalization

this unifies near-Siegel and near-Parabolic renorm theories

key tool: almost-invariant pseudo-Siegel disks

allowing to ``hide'' non-JLC phenomenon

Siegel renorm theory is ``redeveloped'' for
pseudo-Siegel disks in the near-degenerate regime

Inou-Shishikura sectorial bounds are then obtained by
analyzing the inner geometry of pseudo-Siegel disks

one of the new ideas in DL22 is that degenerations can be ``accounted'' on deeper renorm levels (more later)

this motivated:

DD, Lyubich 2023: MLC holds at Feigenbaum points

(on a technical level DL22
and DL23 are quite different)

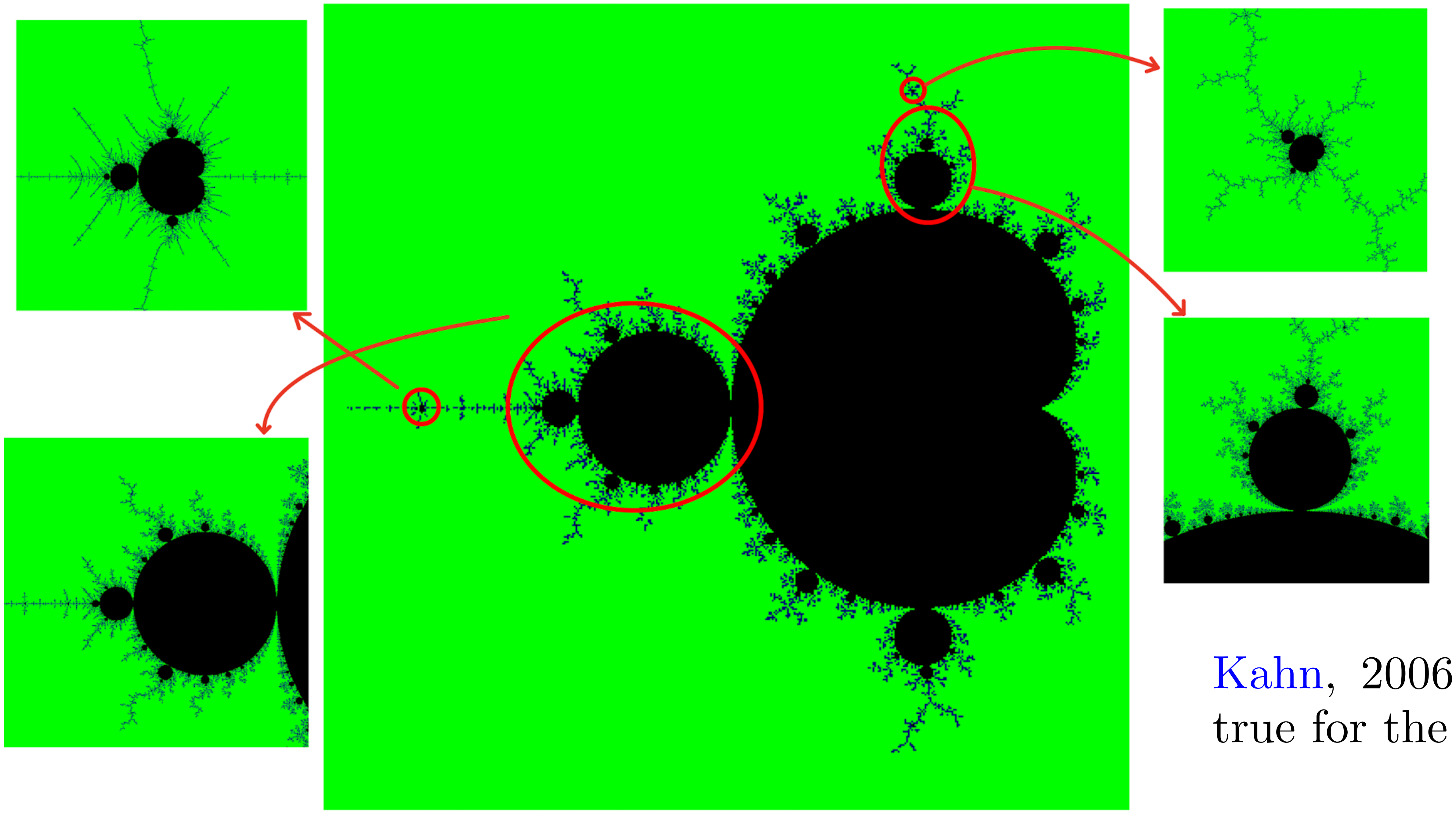
DD, Lyubich 2023: MLC holds at Feigenbaum points

let $\mathbf{R}_i : \mathcal{M}_i \to \mathcal{M}$ be the canonical homeomorphism from a small copy $\mathcal{M}_i$ to the Mandelbrot set $\mathcal{M}$

then $\bigcap_{n\geq 0} \mathbf{R}^{-n}(\mathcal{M}) = \{c_i\}$ is a singleton

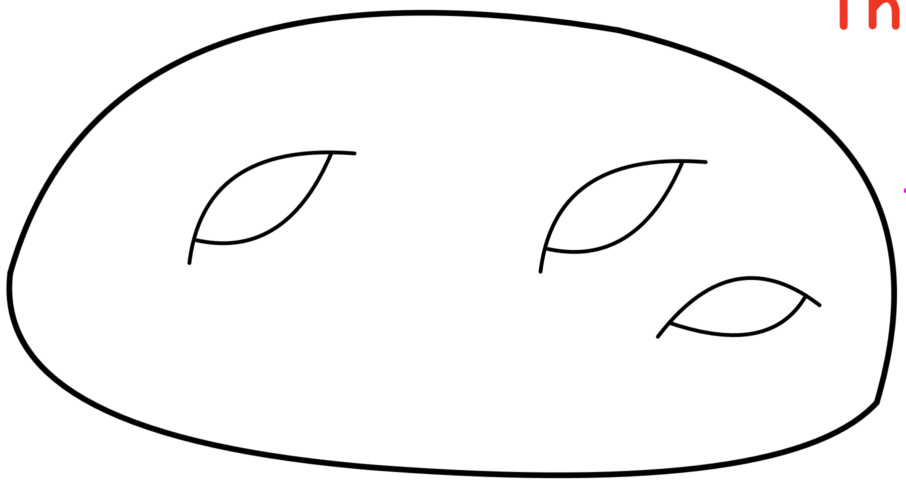
and the MLC holds at c_i

c_i is called a Feigenbaum point

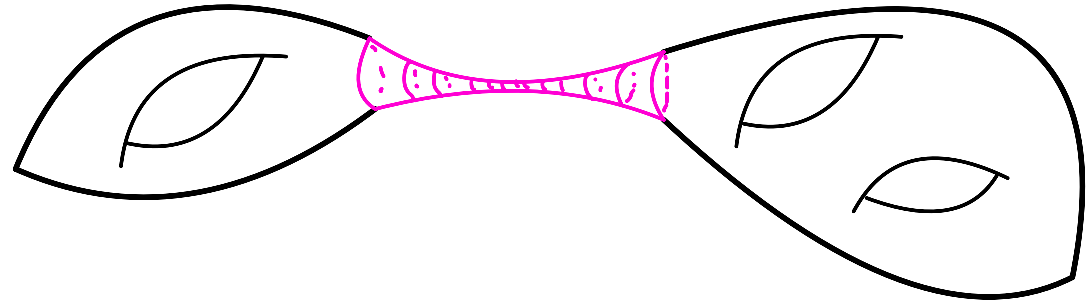


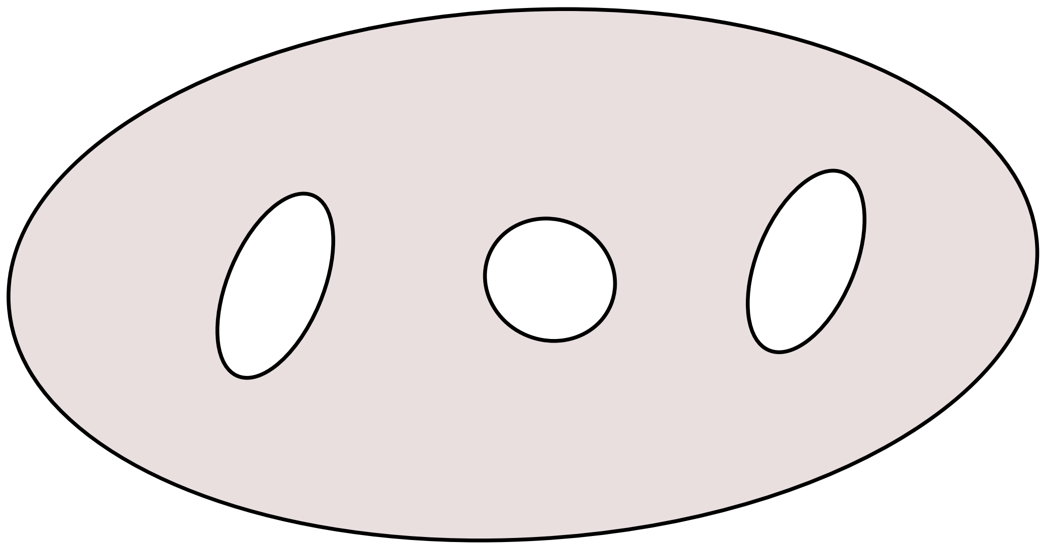
Kahn, 2006 :
true for the primitive $\mathcal{M}_i$

Thin-Thick decomposition:

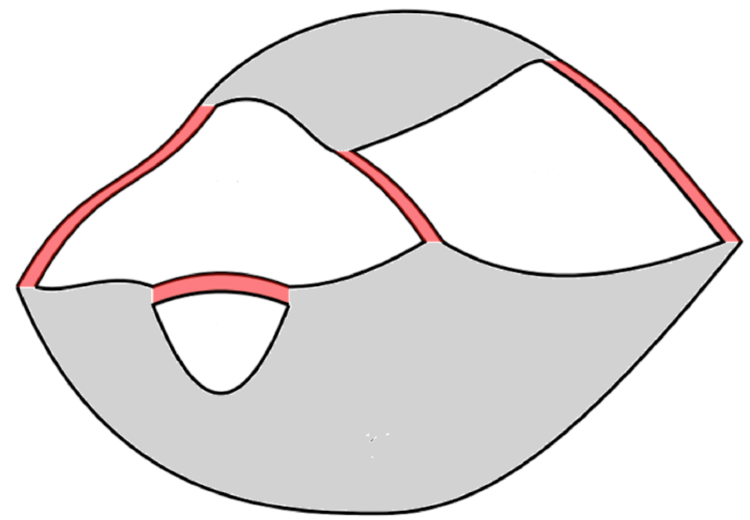


compact surfaces
degenerate along
thin annuli



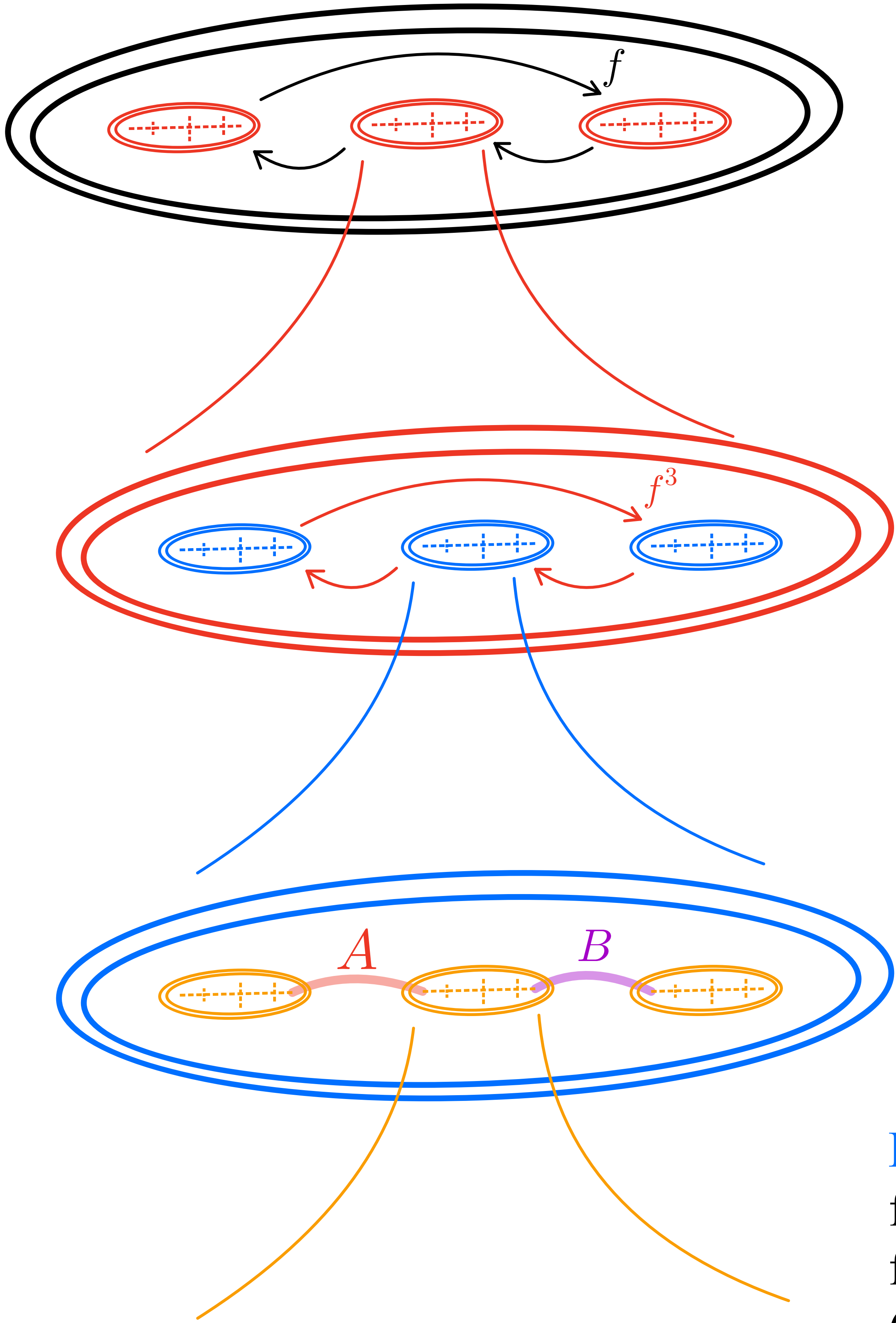


open sets in $\mathbb{C}$
degenerate along
wide rectangles



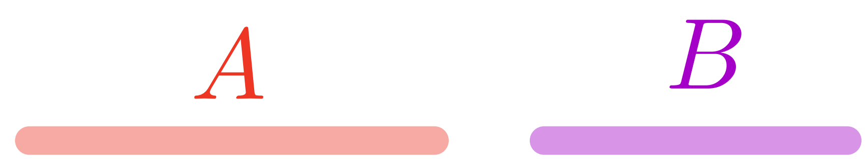
Kahn:
near-degenerate regime for
renormalization theory
of quadratic polynomials

W. Thurston (for 3-manifolds):
compactness results are amenable
for near-degenerate surfaces



Kahn's argument, 2006,
simplified; it says the following
(the airplane combinatorics, for illustration)

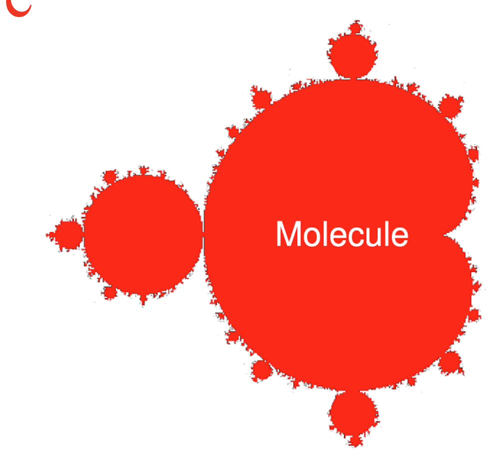
if a degeneration is developed,
then it is formed by
invariant wide rectnalges A, B
alligned with the Hubbard tree

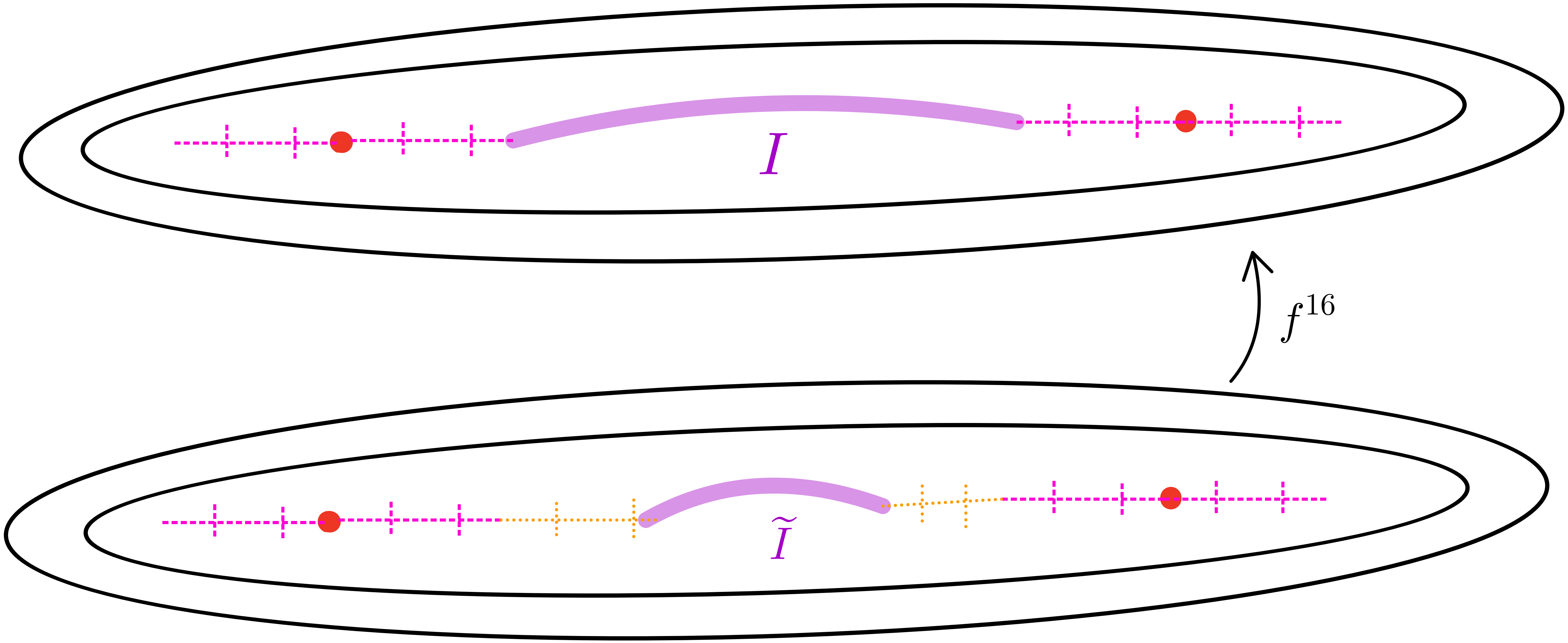


but $A \xrightarrow{f} A \sqcup B$
$B \xrightarrow{f} A$
contradiction!!

i.e., it is based on the fact that the core entropy
of primitive PCF maps is positive

Kahn-Lyubich: MLC holds
for combinatorics ``ε-away"
from the main molecule
(i.e., where core entropy > ε)





DD, Lyubich 2023, simplified:
(the Feigenbaum combinatorics, for illustration)

if Kahn′s argument fails, then we obtain an invariant rectangle I that efficiently overflows its lift $\widetilde{I}$.

The "difference" $I \setminus \widetilde{I}$ consists of two much wider rectangles that hit preperiodic Julia sets of next level.

I. e., if a degeneration emerges, then it starts to increase with super exponential speed.

It is a contradiction to the Teichmüller contraction.