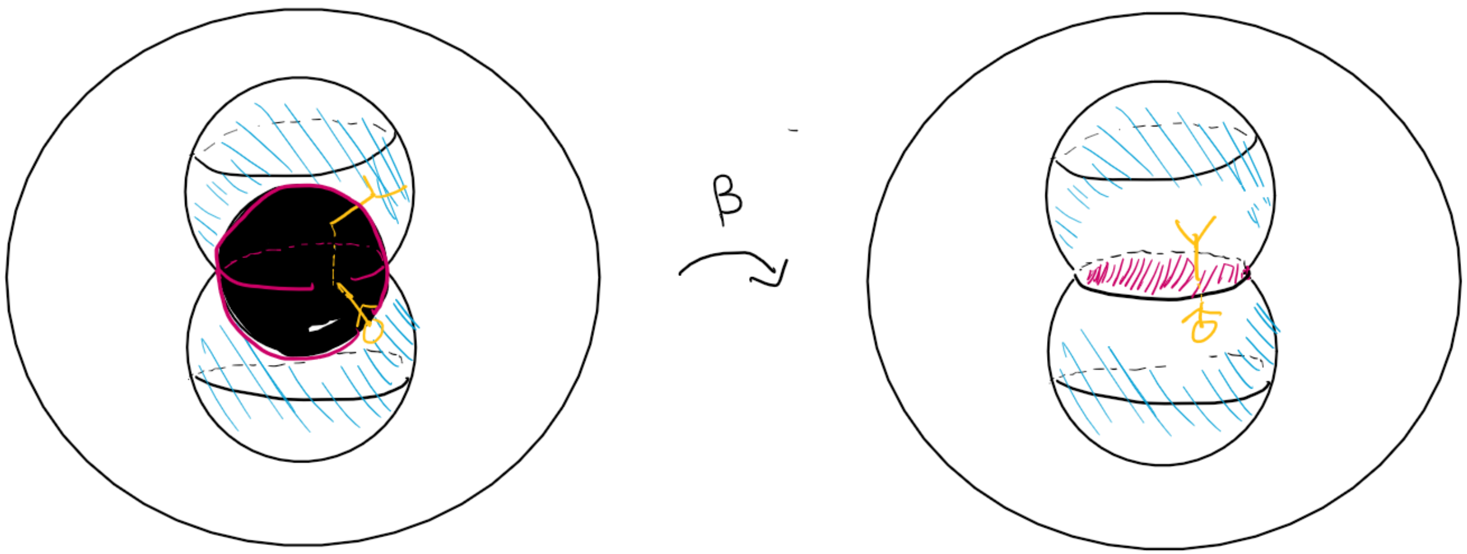
Interpolating Quasiregular Mappings

Jack Burkart, Bard College at Simon's Rock



University of Wisconsin-Milwaukee
AMS Spring Sectional Central Meeting — April 20-21, 2024
Three dimensional work joint with Dan Nicks and Alastair Fletcher
Two dimensional work joint with Kirill Lazebnik

It's a particularly special privilege to speak here today!

My parents, Dave and Hayley, University of Wisconsin-Milwaukee class of 1988 and 1987!

Goal: Outline a new technique that interpolates between quasiregular power mappings of different degrees, and convince you that it can be used to construct quasiregular mappings with interesting dynamics!

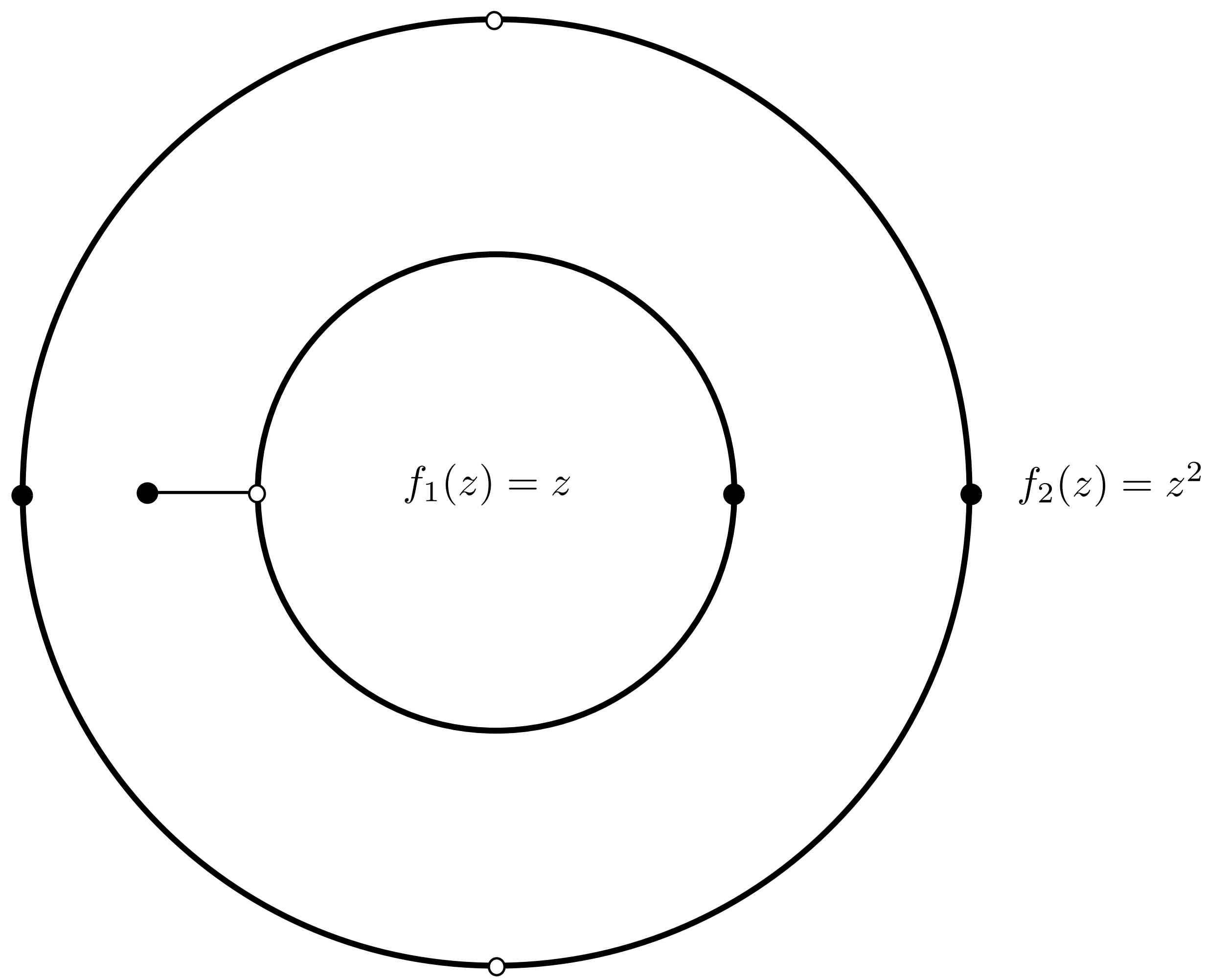
Outline:

1. Motivation: Two dimensional version of this result.
2. Sketching the three dimensional version
3. Things we know we can do
4. Things we *think* we can do.

Model Question: Find a function $f : \mathbb{C} \to \mathbb{C}$ with the following properties

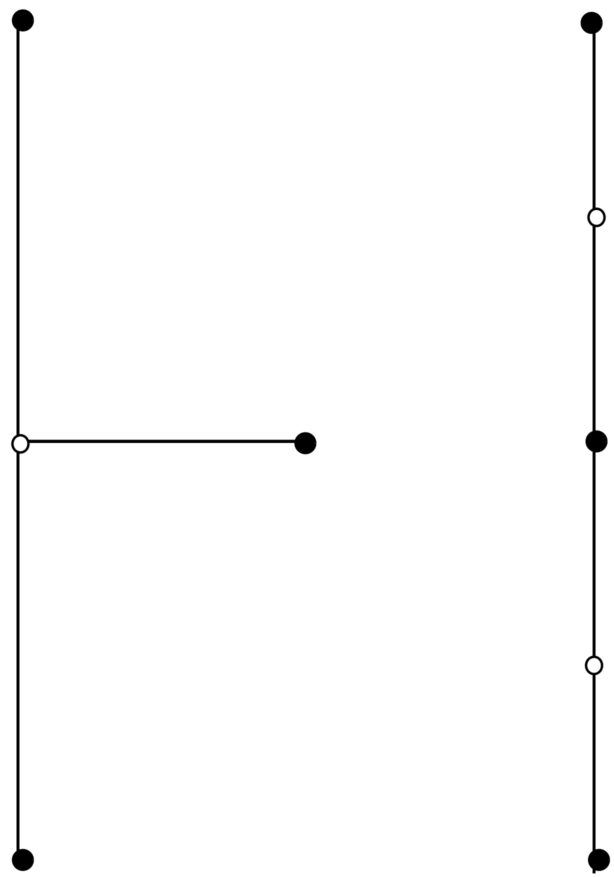
1. $f(z) = z$ on $B(0, 1)$
2. $f(z) = z^2$ on $\mathbb{C} \setminus B(0, r)$ some $r > 1$.
3. $f : \mathbb{C} \to \mathbb{C}$ is quasiregular.

Step 1: Add an antenna to the white vertex.

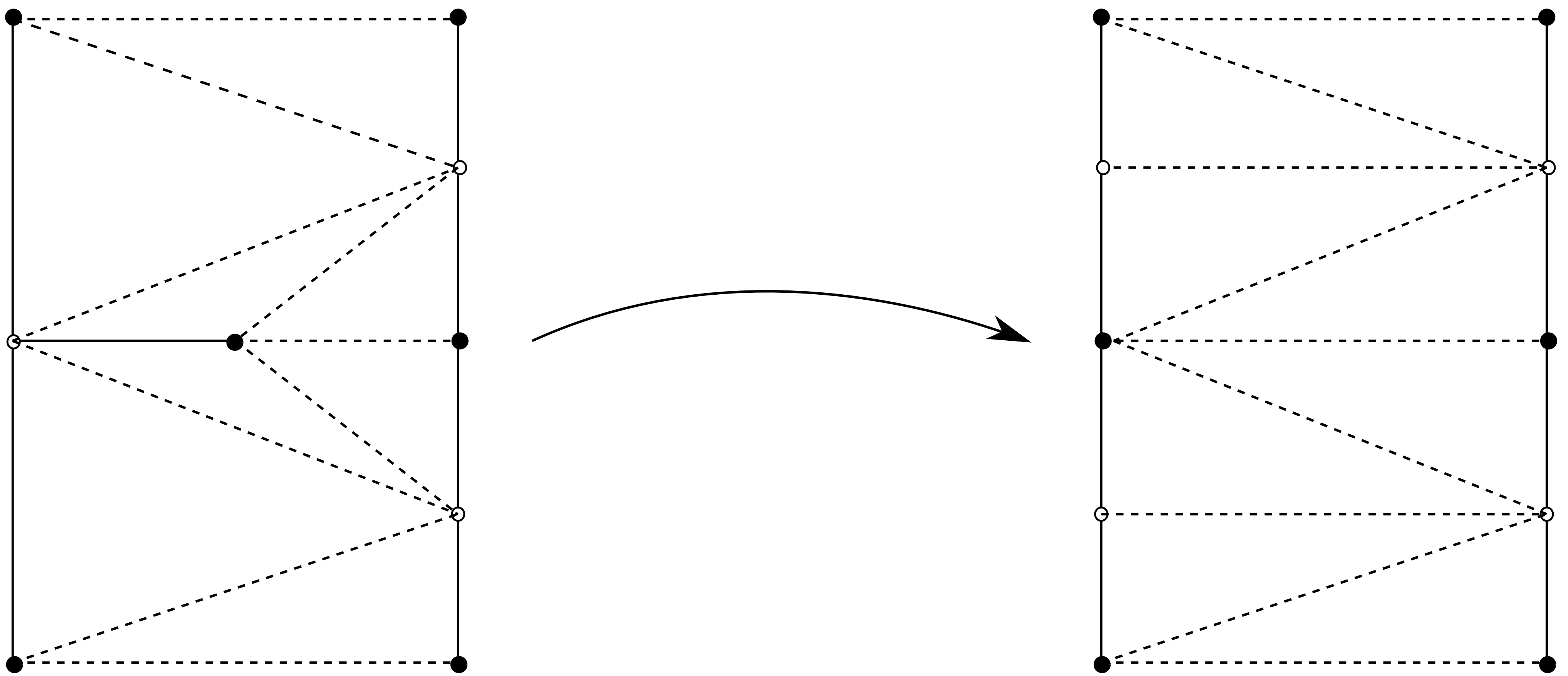


White dots map to negative number, black dots map to positive number.

Step 2: Change coordinates using the logarithm and cut open to a rectangle. We'll glue it back together soon.

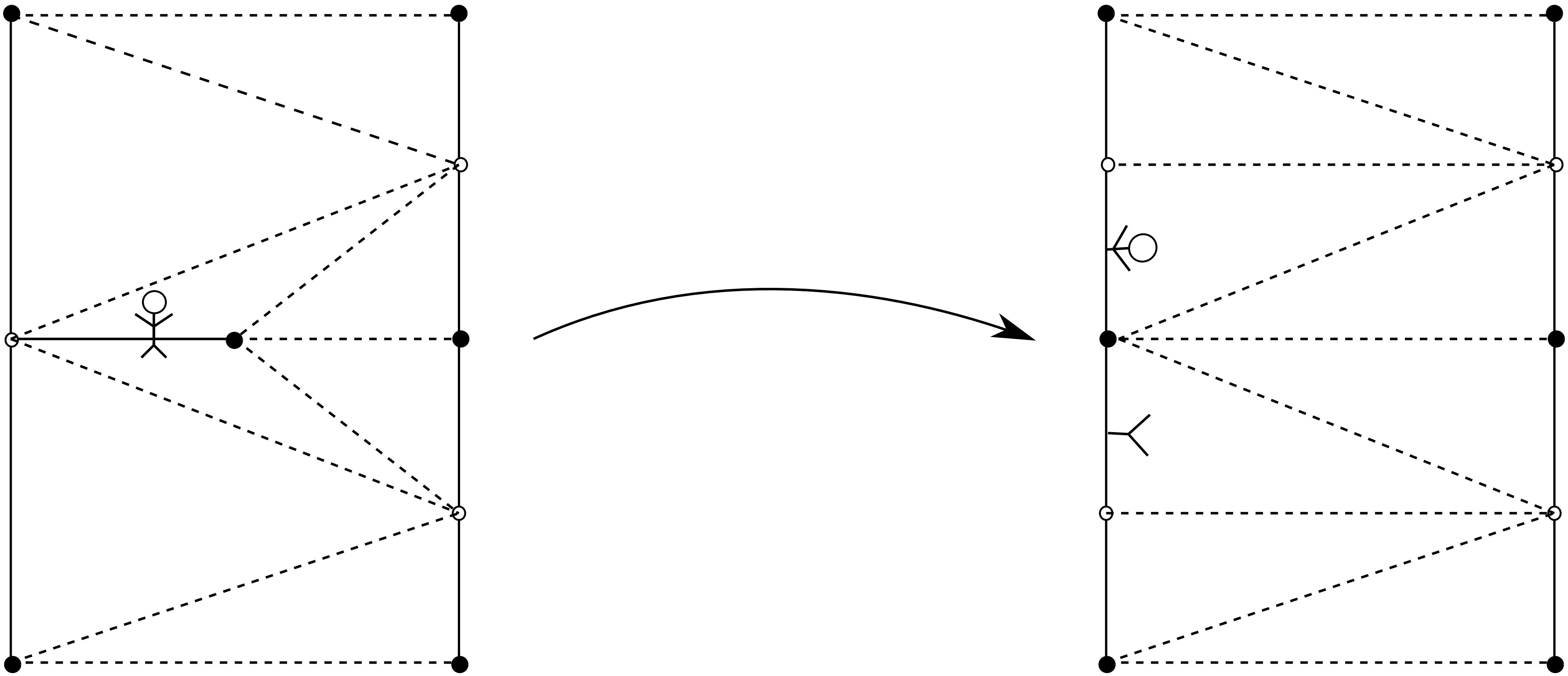


Step 3: Triangulate, and unzip!



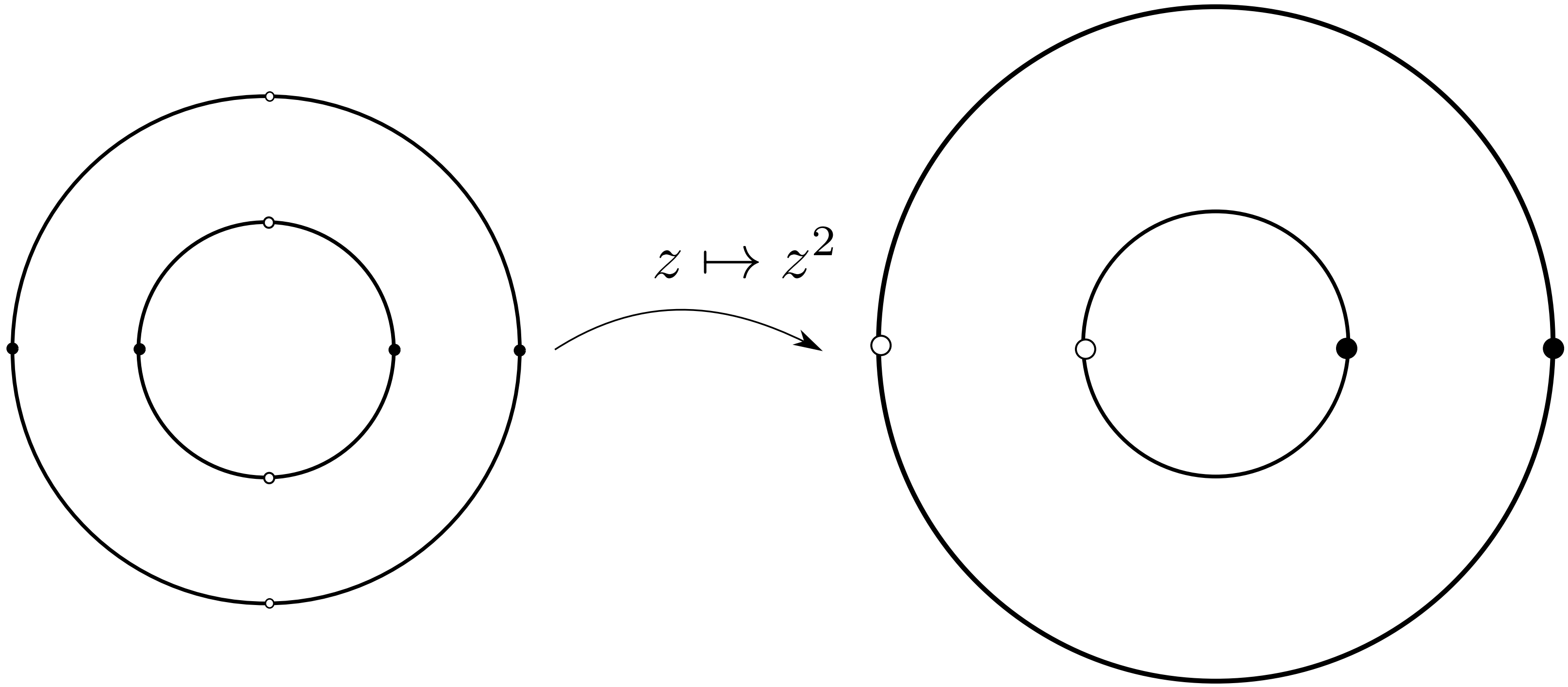
Mapping is piecewise linear on each triangle. What is the problem with this?

Step 3: Triangulate, and unzip!



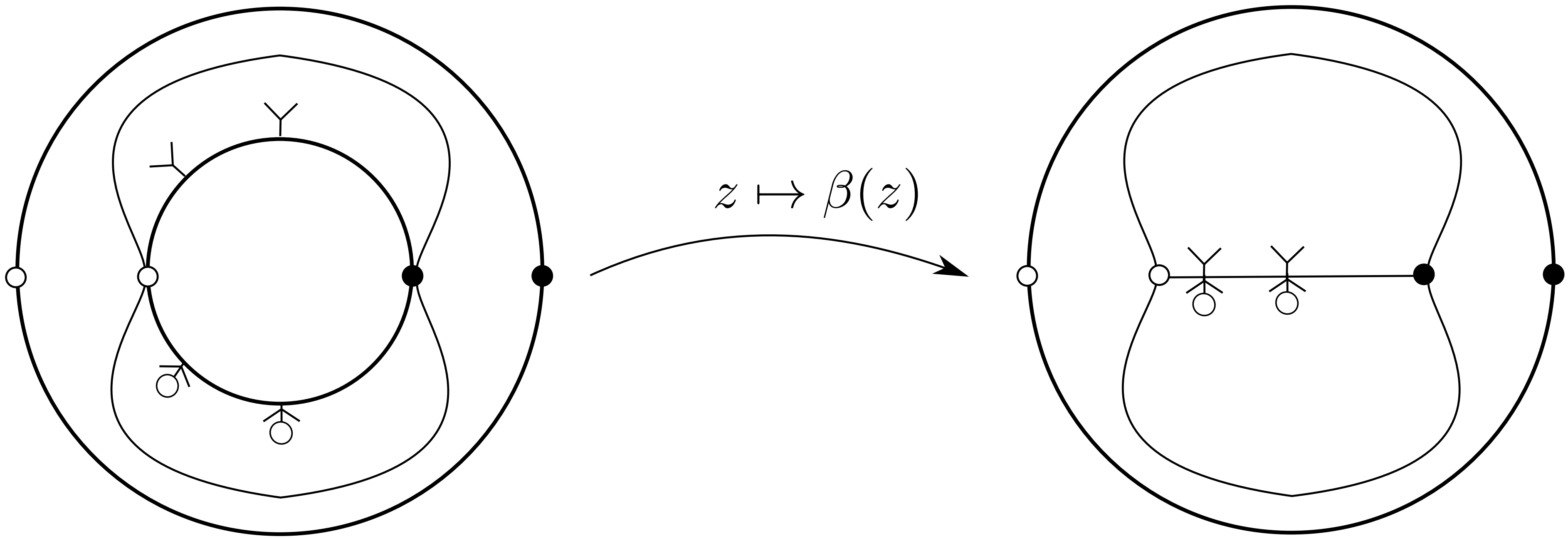
The map cannot possibly be continuous over the antenna!

Step 4: We'll fix it later: Go back and apply $f_2(z) = z^2$.



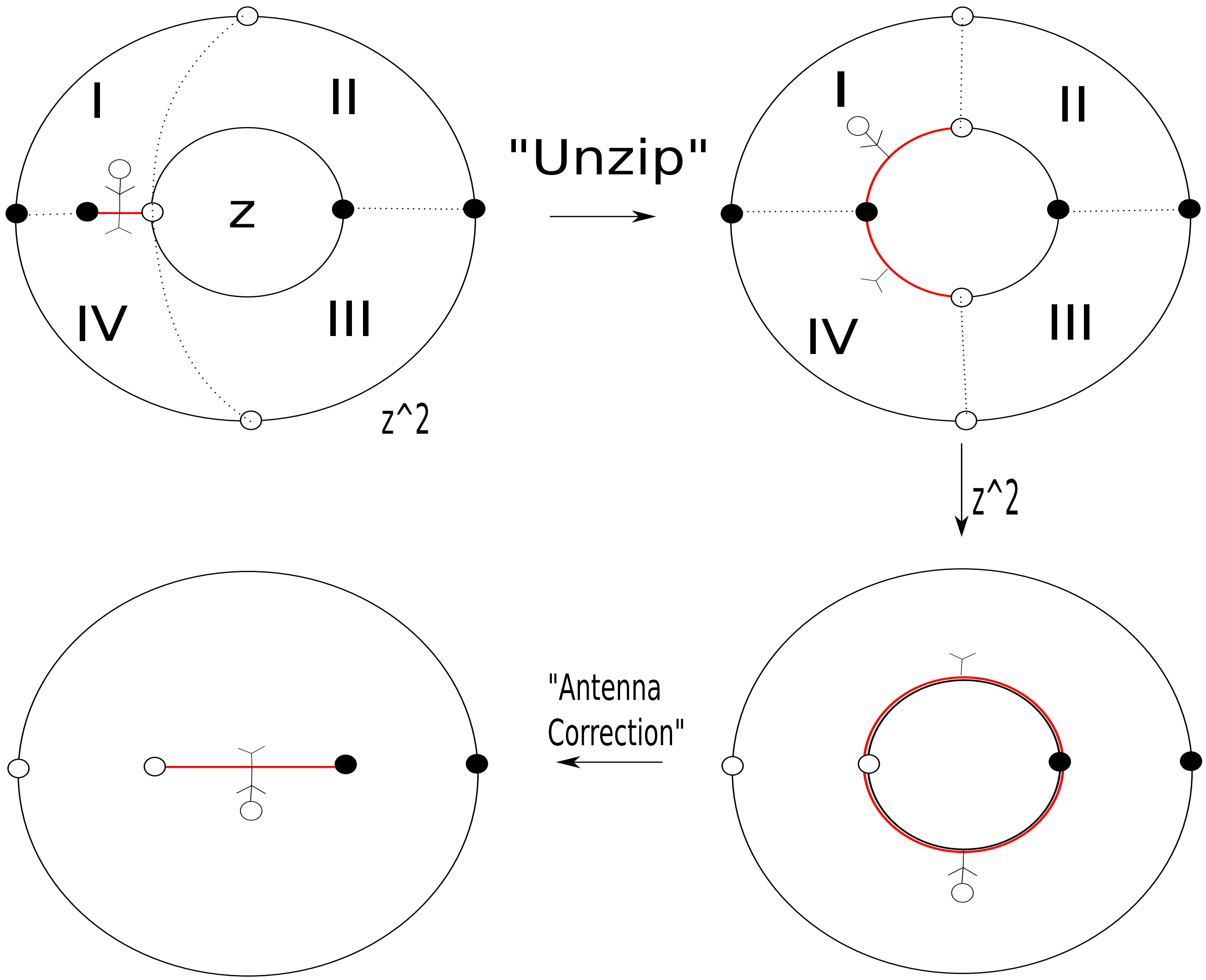
What happened to the stick man?

Step 5: Interpolate identity and circle collapse to glue stick man together.

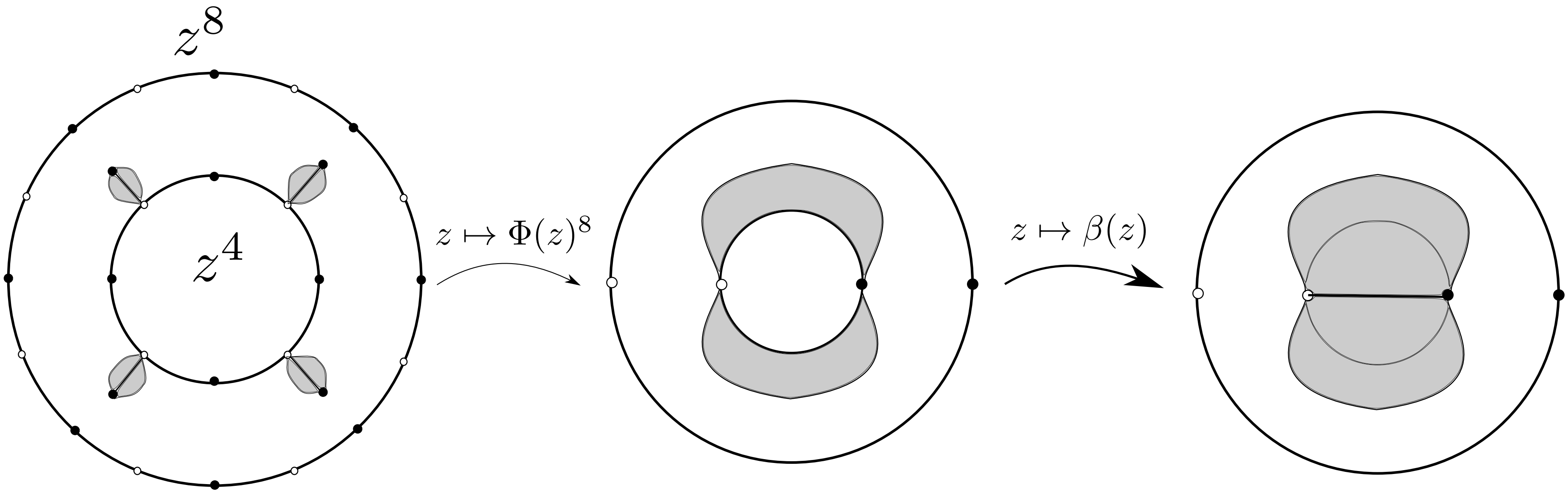


We call this map β. We're done!

Summary: If we do the antenna correction, we get a nice map that interpolates between $f_1(z) = z$ and $f_2(z) = z^2$.

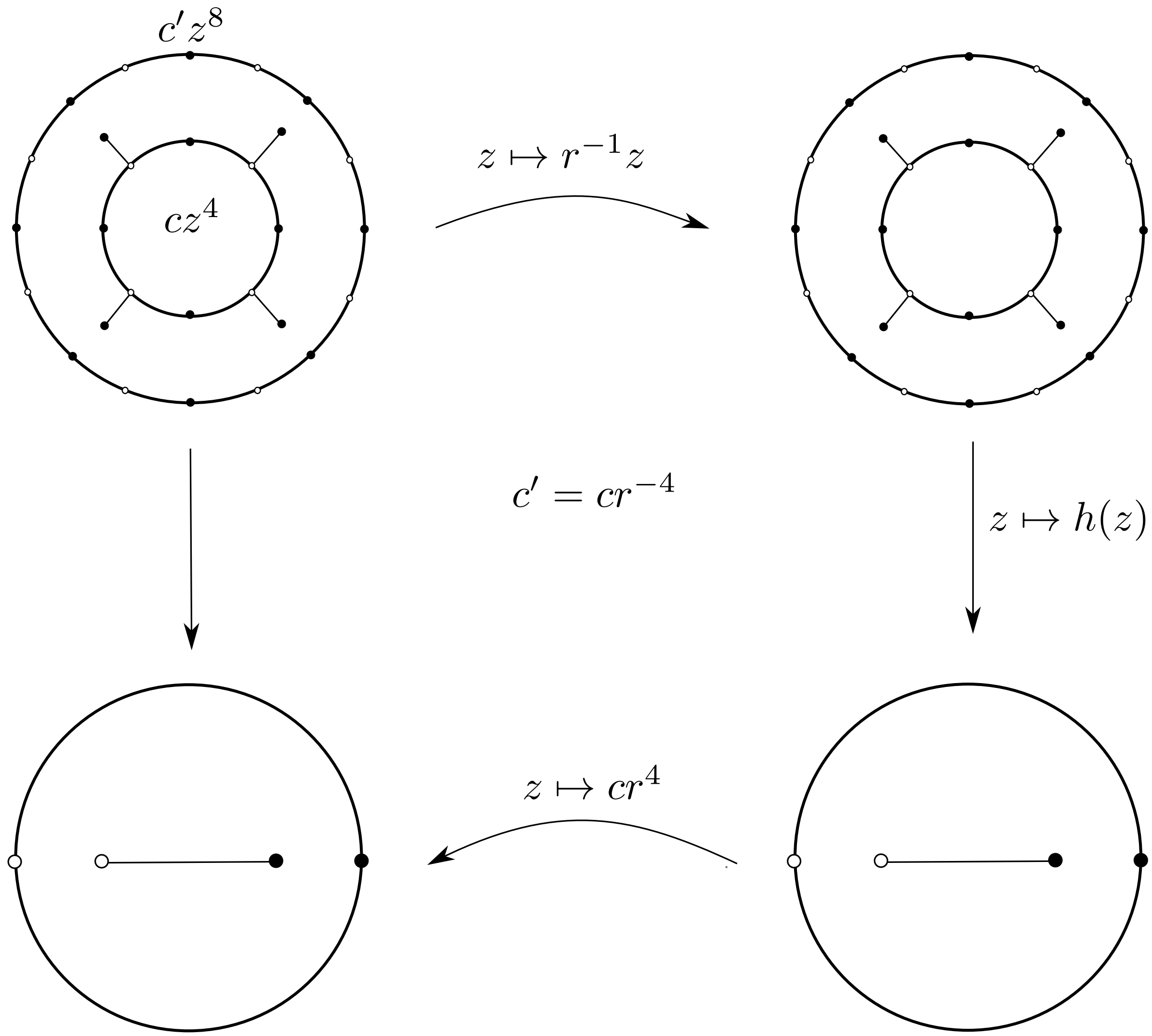


We can perform a similar idea to transition from z^4 to z^8.



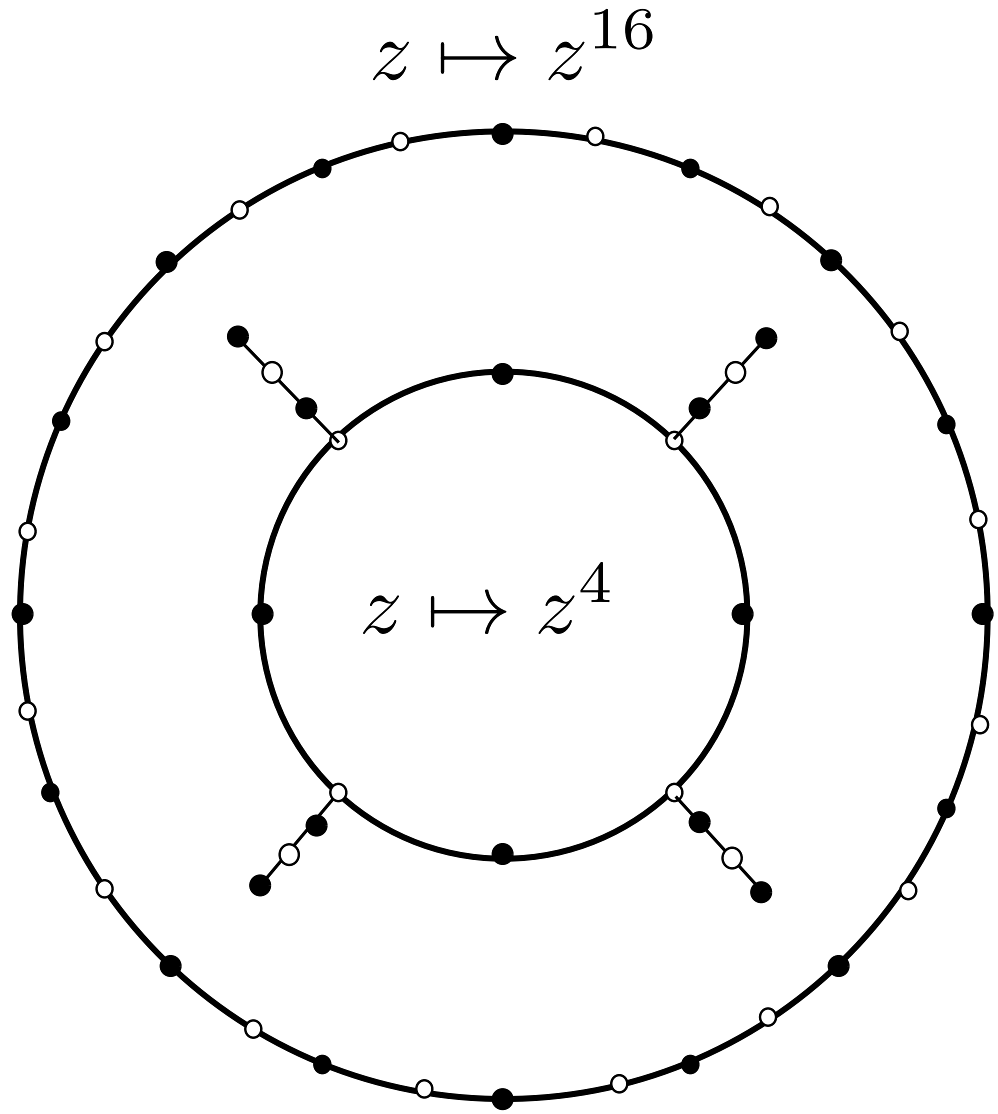
One just adds more antenna, and follows a similar line of reasoning as before.

Interpolation at different radii if coefficients are chosen correctly

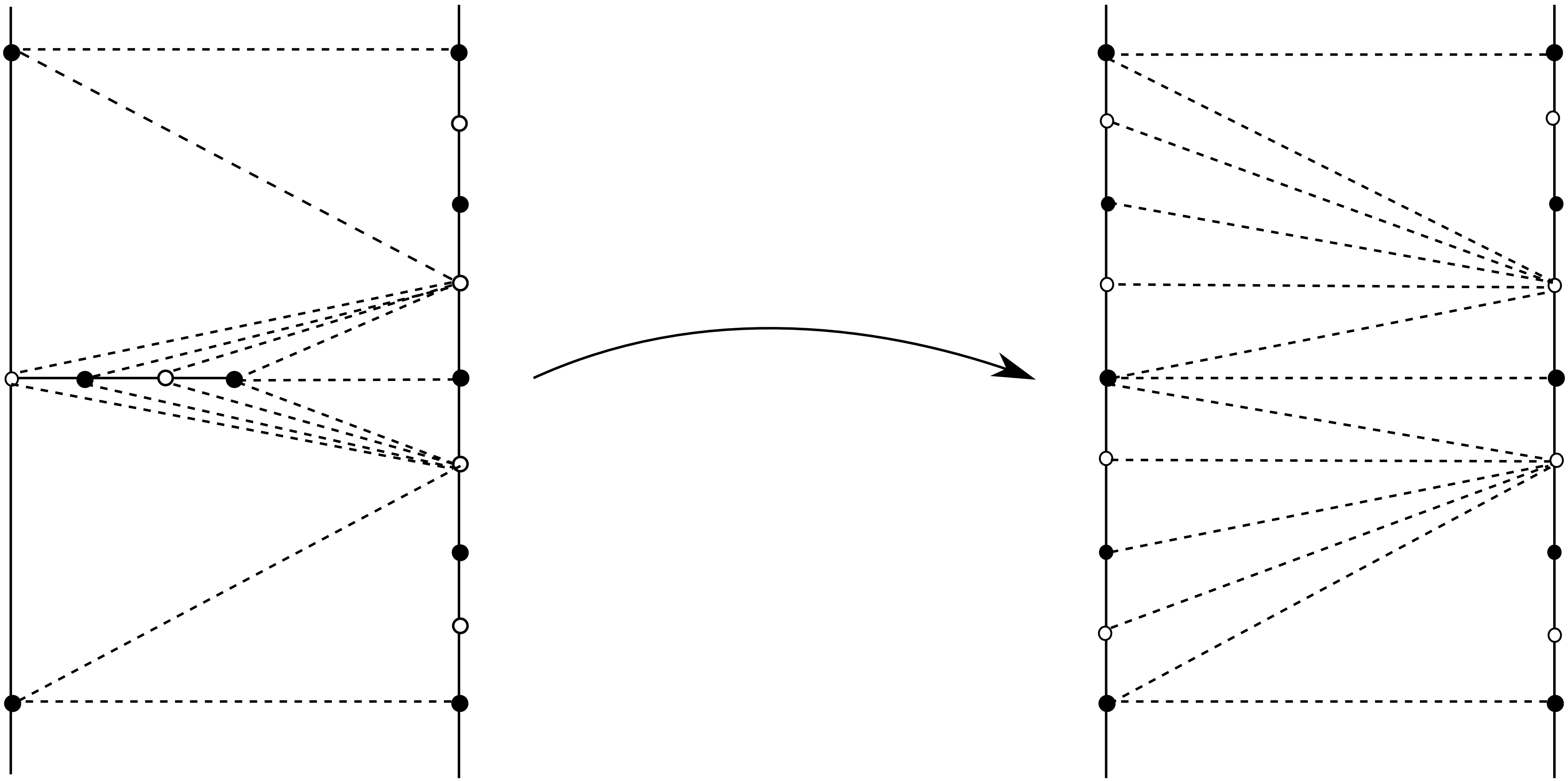


Upshot: Therefore we can apply this interpolation many times!

We can interpolate between larger degrees too.



What Changes? The triangulation part is a bit more complex



As the ratio of the degrees gets large, this triangulation gets very distorted.

What were the key ingredients?

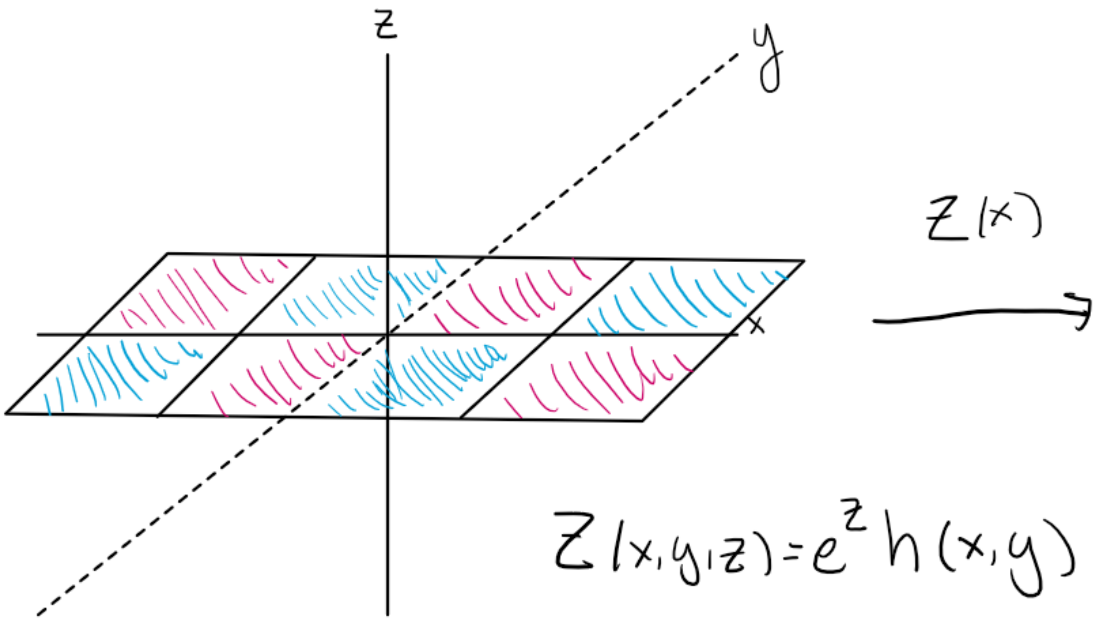
1. Monomials z^n on $\mathbb{C}$
2. Adding antenna to annuli
3. log change of coordinates to cylinder
4. Triangulation
5. "Unfolding" piecewise linear map.
6. A correction map that interpolates between the identity and the Joukowski transformation.

What were the key ingredients?

1. Monomials z^n on $\mathbb{C}$ – **Quasiregular Power Mappings!**
2. Adding antenna to annuli – **Adding 2D Flaps to Spheres**
3. log change of coordinates to cylinder – **The Zorich Map!**
4. Triangulation – **Works perfectly well in higher dimensions**
5. Carefully "unfolding" piecewise linear map – **Again, works perfectly well in higher dimensions**
6. A correction map that interpolates between the identity and the Joukowski transformation – **The "Burger" map**

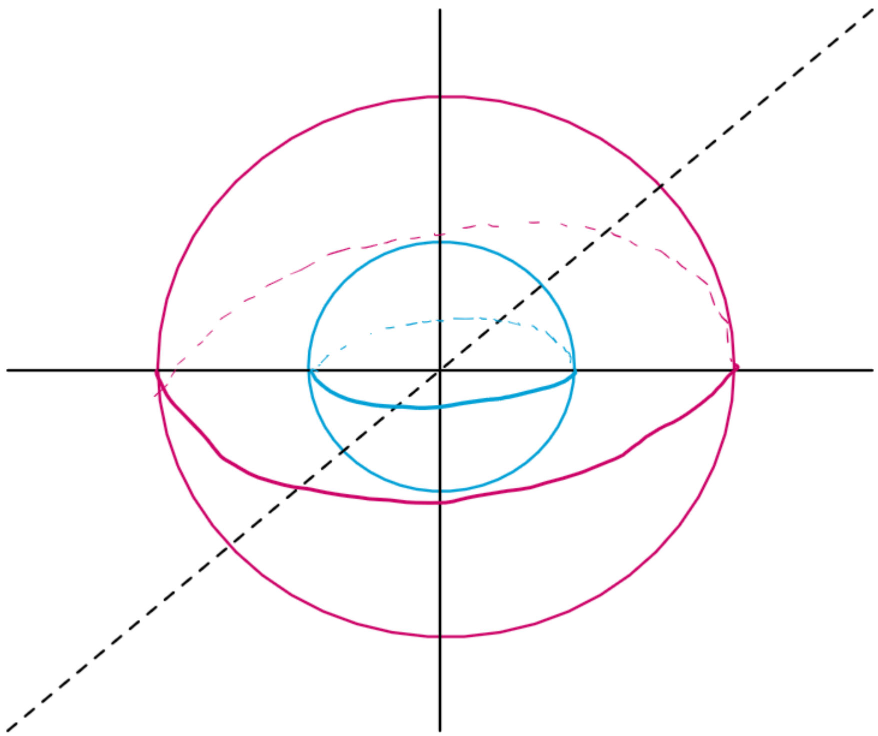
Zorich Map - Key Features

1. $Z : [0,1] \times \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}$
2. Subsquares $[0,1] \times \{r\}$ get mapped to hemispheres of radius $\exp(r)$.
3. Can be extended by reflection and periodicity.
4. Zorich Mappings are used to define analogies to power mappings in the complex plane. We use the formula $P_d(x) = Z(dZ^{-1}(x))$.

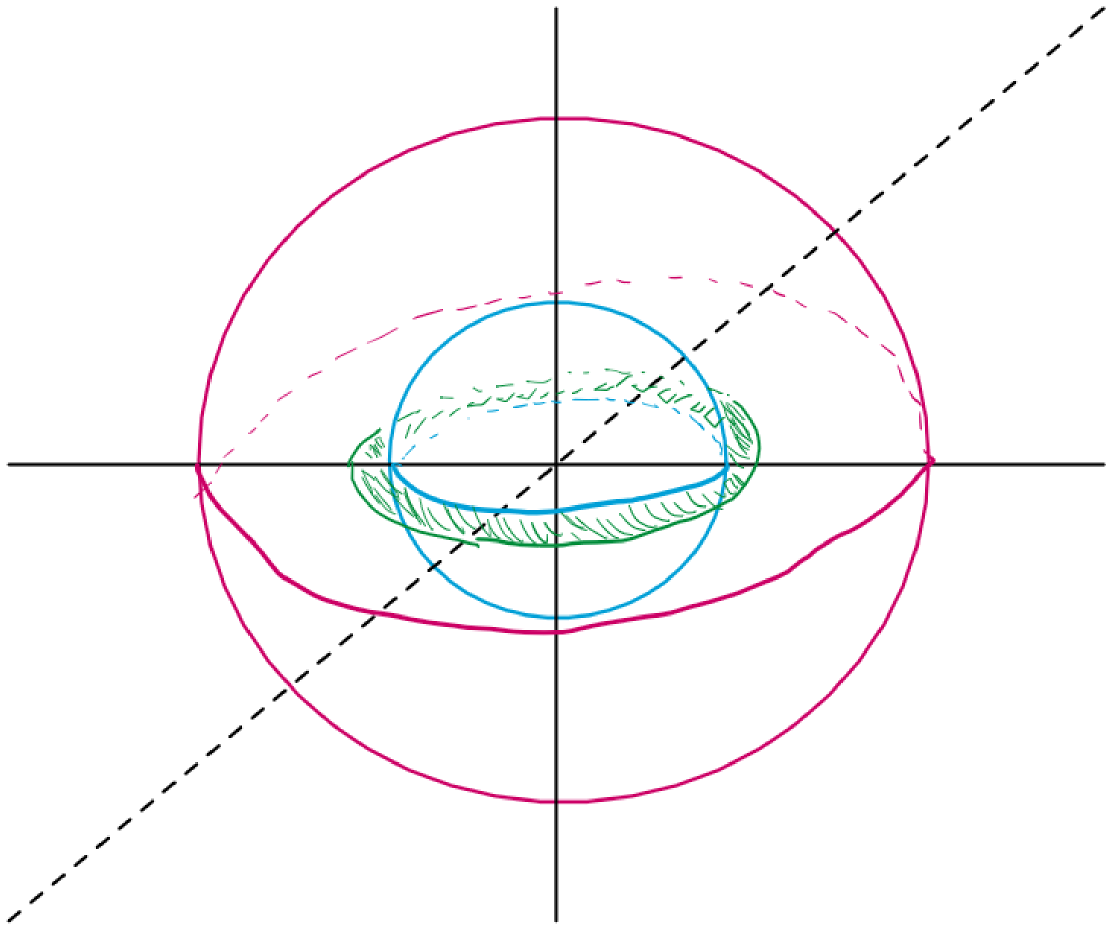


Model Question: Find a function $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

1. $f(z) = P_1(z)$ on $B(0, 1)$
2. $f(z) = P_3(z)$ on $\mathbb{R}^3 \setminus B(0, r)$ some $r > 1$.
3. $f : \mathbb{R}^3 \to \mathbb{R}^3$ is quasiregular.

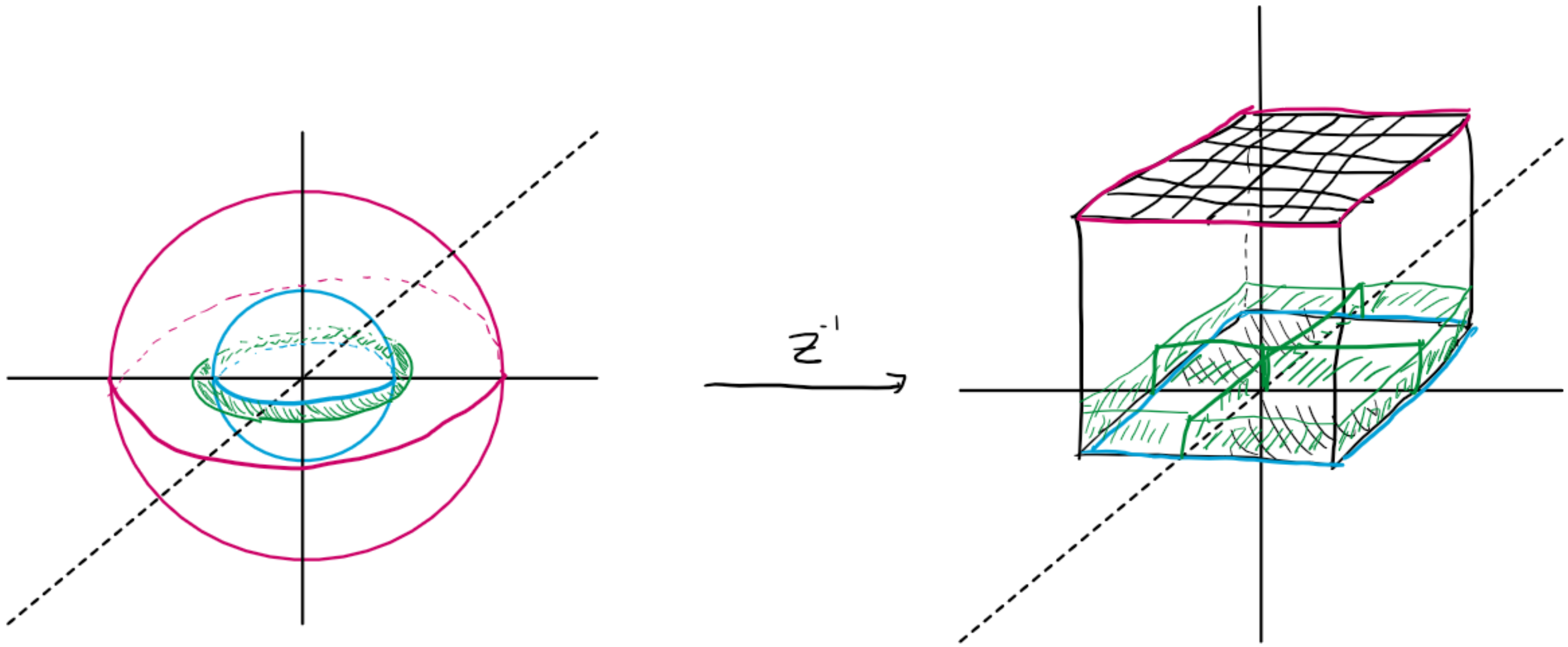


Step 1: Add **flaps** to the sphere,

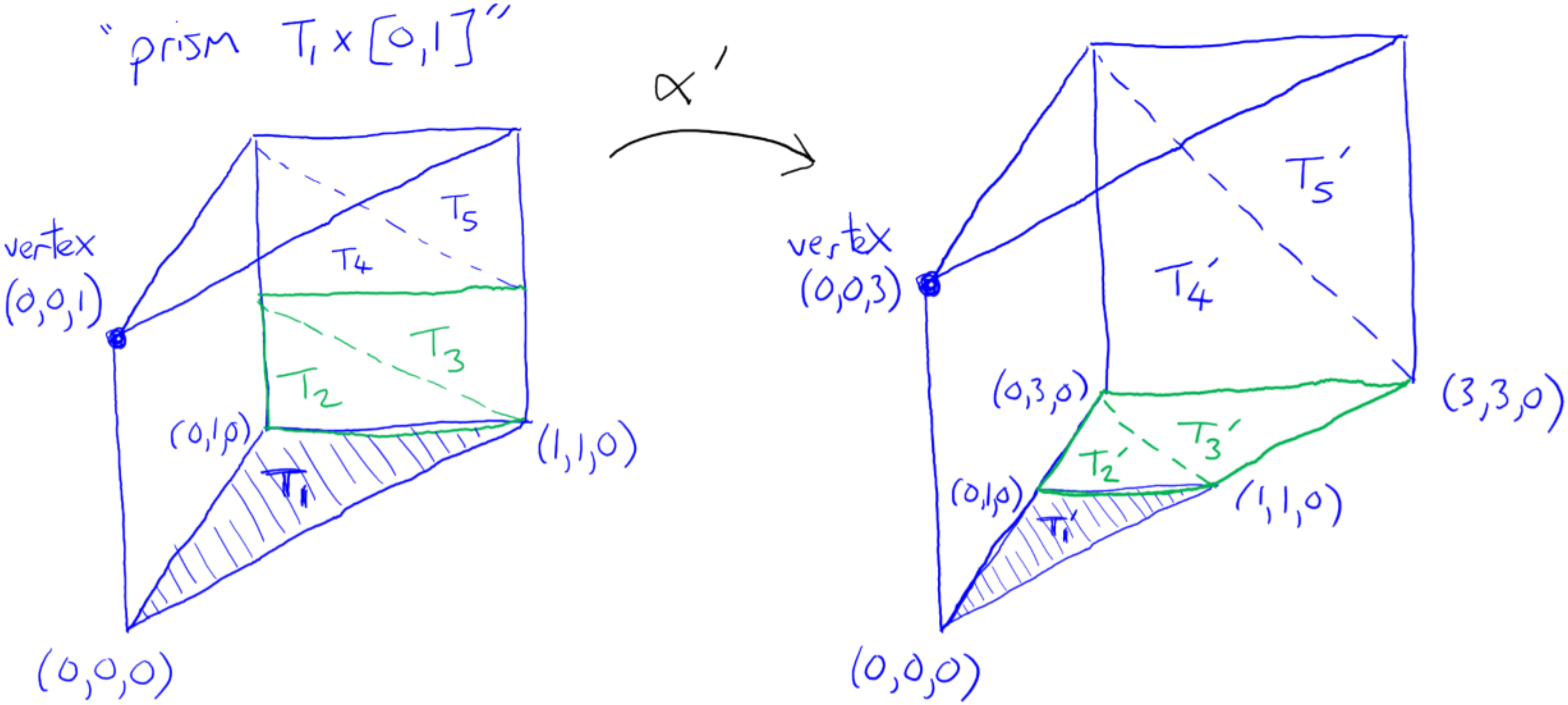


The green region is rectangular flap.

Step 2: Change Coordinates using **The Zorich Mapping** to do the unfolding step.

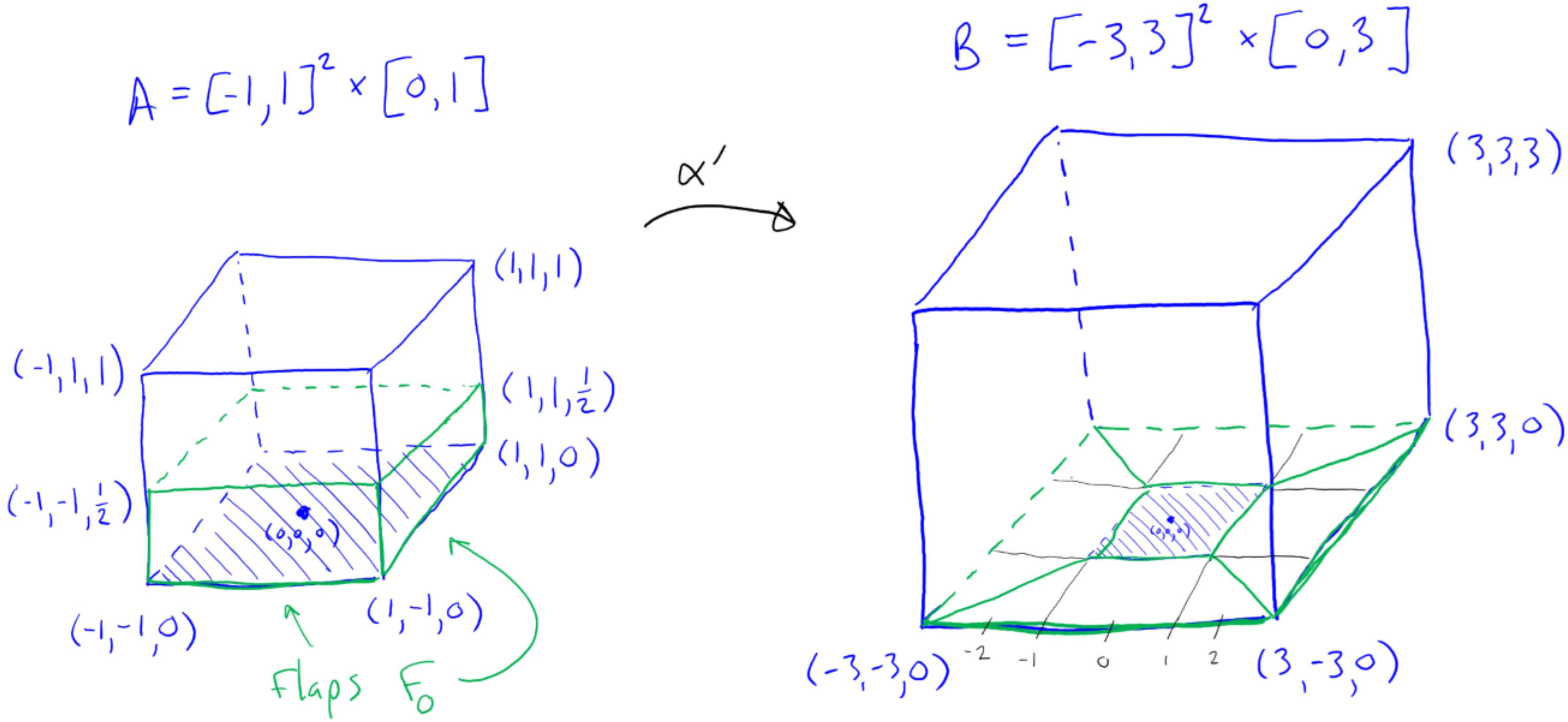


Step 3: Triangulate, and do some origami/unfolding.



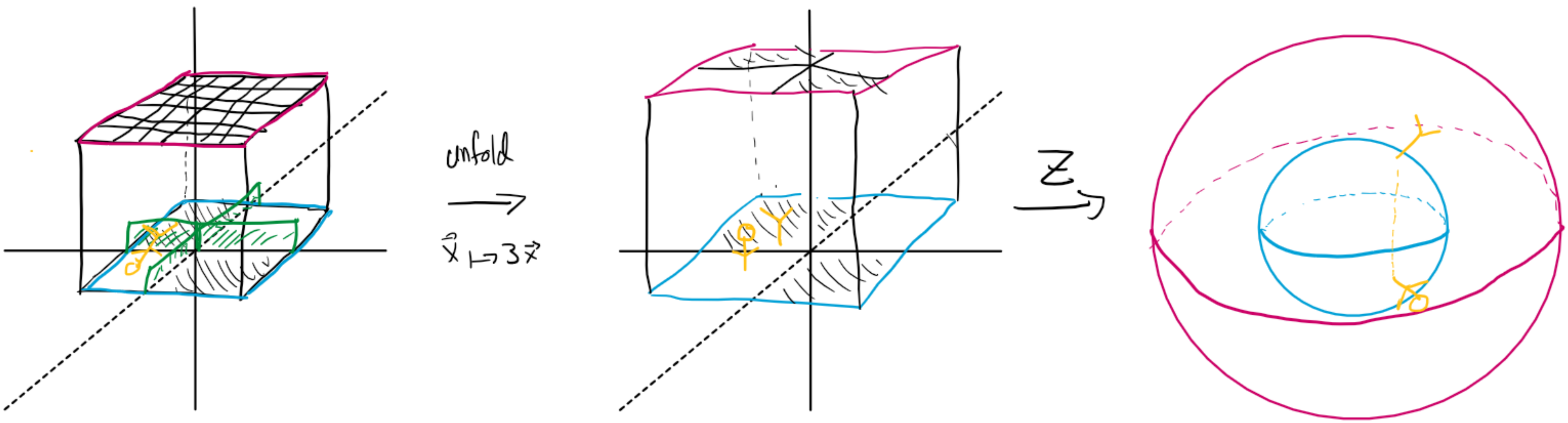
One of 8 prisms where we define the unfolding.

Step 3: Triangulate, and do some origami/unfolding.



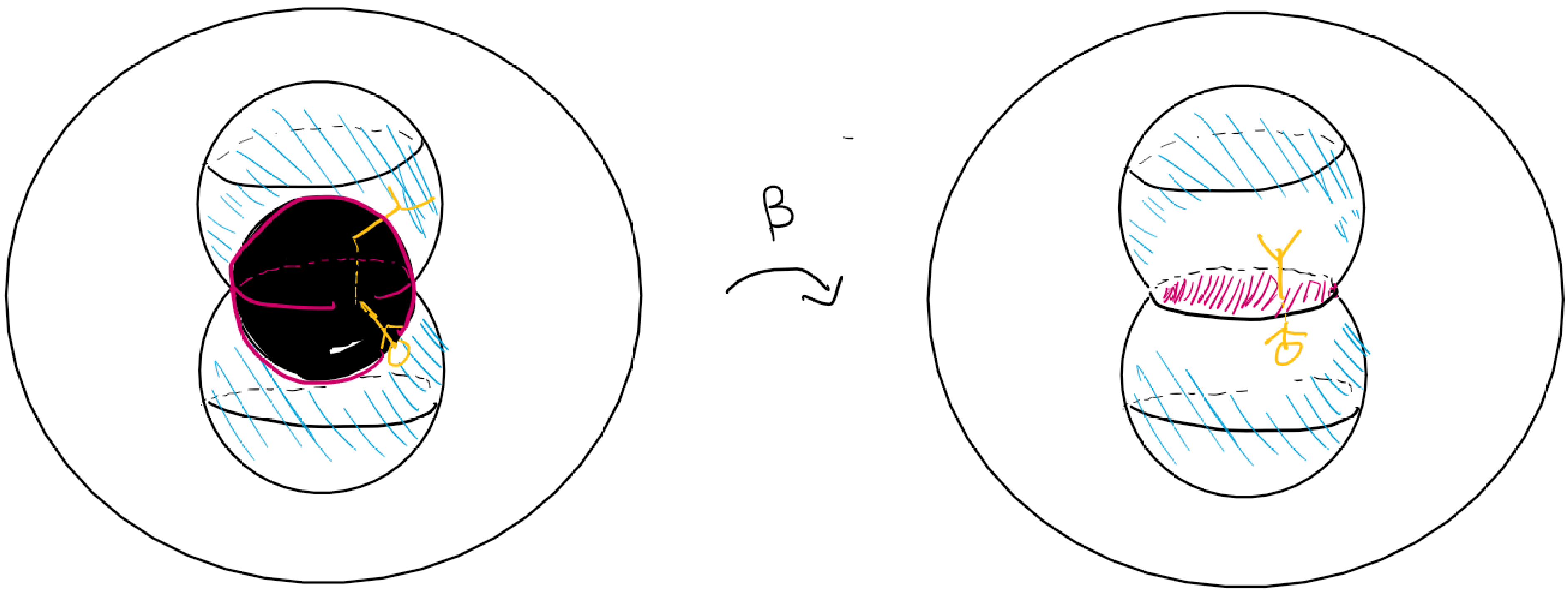
The result in one square

Step 4: We'll fix continuity later. In this rectangular prism, we can now complete the process of applying the power map $P_3(z) = Z(3P^{-1}(z))$



After unfolding, the checkerboards align and we can apply $x \mapsto 3x$. Then we can switch back to regular coordinates via the Zorich map.

Step 5: Interpolate identity on a "burger" and collapse the sphere to fix continuity issues. This mapping is called β and is 3^2-quasiconformal.



The domain is the union of the two balls; the sphere of radius one centered at the origin is removed. The sphere is collapsed, fixing the continuity issue.

Just as before, in the aggregate, we get the desired model mapping. We can iterate this construction, creating quasiregular functions of transcendental type.

Applications: The maximum modulus of a function is given by

$$M(r, f) = \max_{|x|=r} |f(x)|$$

The growth rate is

$$\limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

Theorem (B, Fletcher, Nicks): Using this construction above, there exists quasiregular mappings with arbitrarily fast or arbitrarily slow growth.

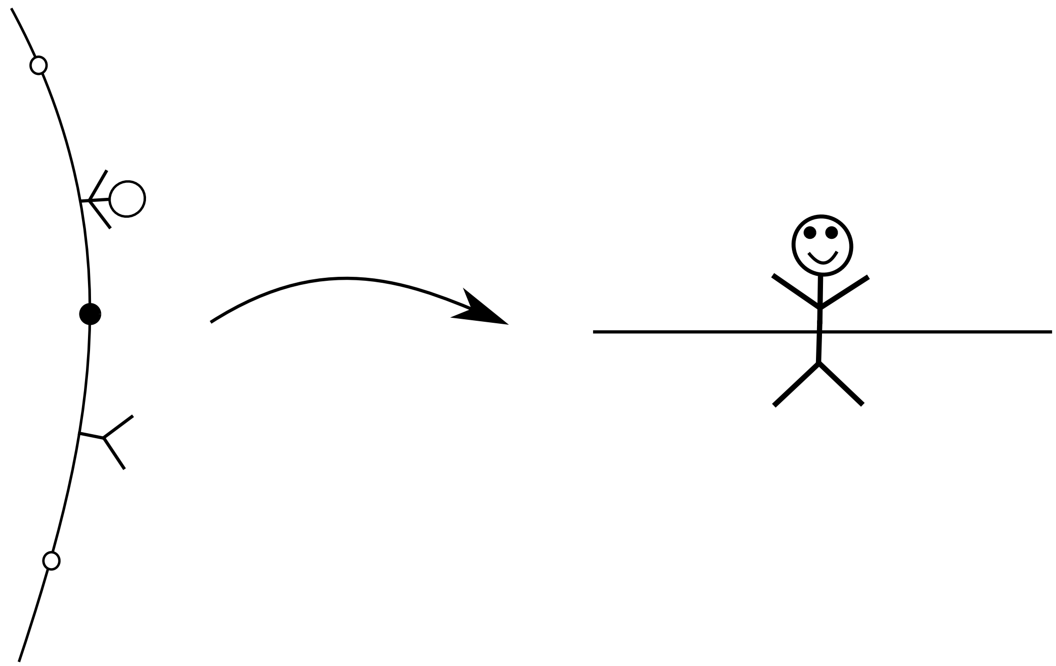
Theorem (B, Fletcher, Nicks): There exists a quasiregular mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ whose outermost and innermost quasi-Fatou components are round spheres. The quasi-Fatou components are hollow and wandering.

I believe this is the first such mapping where there are connected components of the boundary of the quasi-Fatou set that have dim(2).

Question: What is the dimension of the Julia set of the above mapping? Might it also be 2?

We believe this is the case, but the details present challenges.

1. The quasi-Fatou components above have infinitely many bounded complementary components.
2. The insides of those components have infinitely many bounded complementary components
3. ..... and it continues indefinitely
4. In dimensions bigger than 2, one can use a so called conformal elevator to control the distortion on these components. We need to judiciously use clever interpolations to control the distortion. This is technical, but feasible.



Thank You! Any Questions?