Dynamics of the Composition of Quadratic Maps with Functions of Constant Complex Dilatation

by Mark Broderius

We will generalize the functions P_c defined by $P_c(z) = z^2 + c$ by applying transformations by a stretch and a couple of rotations.

Define S_K by $S_K(x + iy) = Kx + iy$. This applies a horizontal stretch by a factor of K.

Conjugating S_K by the rotation function $r_\theta(z) = e^{i\theta}z$ stretches by K in the direction of θ instead.

This results in the function $h_{K,\theta}$ defined by $h_{K,\theta}(z) = r_\theta \circ S_K \circ r_{-\theta}(z)$.

Define $H_{K,\theta,c}$ by $H_{K,\theta,c}(z) = [h_{K,\theta}(z)]^2 + c$. This function is quasiregular.

Note that $H_{1,\theta,c}(z) = P_c(z) = z^2 + c$.

For any fixed point z_0 of $H_{K,\theta,c}(z)$, we can determine the local behavior of $H_{K,\theta,c}$ around z_0 by using a Jacobian matrix and classifying its eigenvalues λ_1 and λ_2.

This is the matrix.

$$\begin{bmatrix} \frac{\partial(\Re(H_{K,\theta,c}(z_0)))}{\partial x} & \frac{\partial(\Re(H_{K,\theta,c}(z_0)))}{\partial y} \\ \frac{\partial(\Re(H_{K,\theta,c}(z_0)))}{\partial x} & \frac{\partial(\Im(H_{K,\theta,c}(z_0)))}{\partial y} \end{bmatrix}$$

1. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then z_0 is an attracting fixed point.

2. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then z_0 is a repelling fixed point.

3. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$, then z_0 is a saddle fixed point.

$I(H_{K,\theta,c}) = \{z \in \mathbb{C} : \lim_{n\to\infty} H^n_{K,\theta,c}(z) = \infty\}$
denotes the escaping set of $H_{K,\theta,c}$.

$BO(H_{K,\theta,c})$
$= \{z \in \mathbb{C} : \{|H^n_{K,\theta,c}(z)| : n \in \mathbb{N}\} \text{ is bounded}\}$
denotes the bounded orbit set of $H_{K,\theta,c}$.

These two sets are complements in $\mathbb{C}$.
(There are no bunjee sets in this setting.)

The Mandelbrot set $\mathcal{M}$ consists of the values c such that $0 \in BO(P_c)$, where $P_c(z) = z^2 + c$.

If $c \in \mathcal{M}$, then $BO(P_c)$ is connected. If $c \notin \mathcal{M}$, then $BO(P_c)$ is totally disconnected.

If $c \notin \mathcal{M}$, then P_c cannot have an attracting fixed point since any attracting fixed point necessarily has a basin of attraction.

We can define an analog of the standard Mandelbrot set $\mathcal{M}$ for any fixed K and θ.

Let $\mathcal{M}_{K,\theta} = \{c \in \mathbb{C} : 0 \in BO(H_{K,\theta,c})\}$.

This set can equivalently be characterized as the set of c values such that $\partial I(H_{K,\theta,c})$ is connected.

If $c \notin \mathcal{M}_{K,\theta}$, then $BO(H_{K,\theta,c})$ has uncountably many components.

However, we may NOT necessarily conclude that $BO(H_{K,\theta,c})$ is totally disconnected.

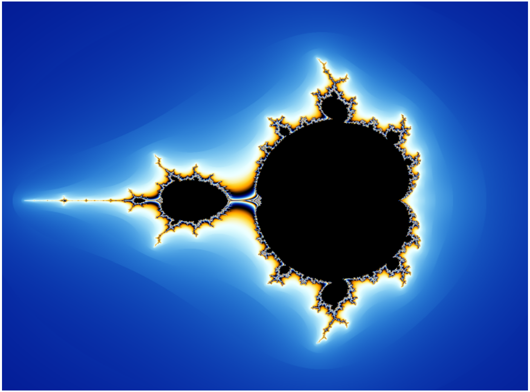

Figure: The Set $\mathcal{M}_{1.25,0.25}$

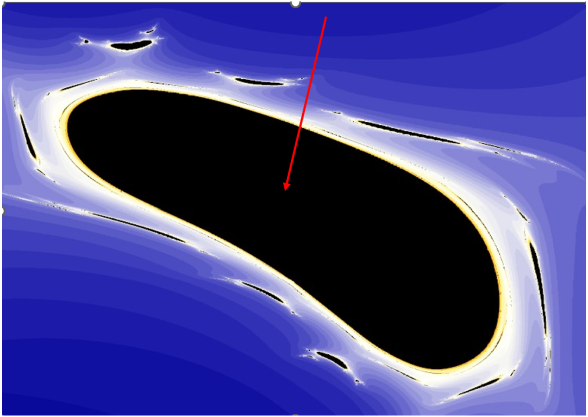

Figure: Attracting Fixed Point of $H_{0.5,0,-1.5-0.5i}$

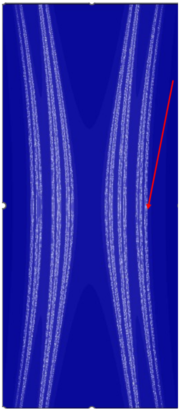

Figure: Saddle Fixed Point of $H_{5,0,-0.1}$

There is an analog of Böttcher's theorem which implies that there is a quasiregular map $\psi_{K,\theta,c}$ that conjugates $H_{K,\theta,c}$ to $H_{K,\theta,0}$ in a neighborhood of infinity.(Fletcher and Fryer)

That is, $\psi_{K,\theta,c} \circ H_{K,\theta,c} \circ \psi_{K,\theta,c}^{-1}(z) = H_{K,\theta,0}(z)$.

This is helpful because $H_{K,\theta,0}$ is star-like about the origin.

We can therefore define a function $\rho_{K,\theta}$ on $[0, 2\pi)$ by setting $\rho_{K,\theta}(\phi)$ equal to the unique element on $\partial I(H_{K,\theta,0})$ in the direction of ϕ.

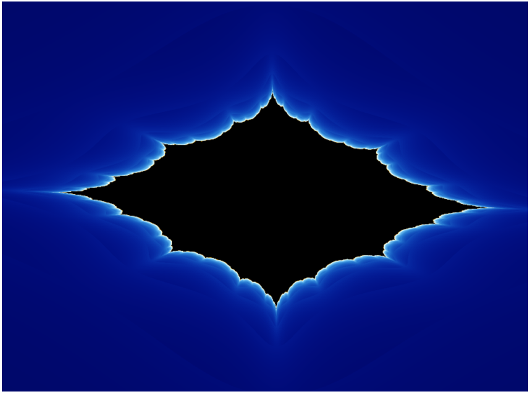

Figure: Picture for $H_{2,\frac{\pi}{2},0}$

We now define a function $\tau_{K,\theta,0}$ from $\mathbb{C}$ to $\mathbb{R}$ by $\tau_{K,\theta,0}(re^{i\phi}) = \frac{r}{|\rho_{K,\theta}(\phi)|}$.

Notably, $\tau_{K,\theta,0}$ sends every point on $\partial I(H_{K,\theta,0})$ to 1.

Also, if s is real, $\tau_{K,\theta,0}(sz) = s\tau_{K,\theta,0}(z)$.

In a neighborhood of infinity, we can define a map $\tau_{K,\theta,c}$ by $\tau_{K,\theta,c}(z) = \tau_{K,\theta,0} \circ \psi_{K,\theta,c}(z)$. Using this definition, we can derive the equation

$$\tau_{K,\theta,c}(H_{K,\theta,c}(z)) = [\tau_{K,\theta,c}(z)]^2$$

.

In other words, $\tau_{K,\theta,c}$ conjugates $H_{K,\theta,c}$ to the map $z \to z^2$. We can therefore continuously extend $\tau_{K,\theta,c}$ to $I(H_{K,\theta,c})$ by setting $\tau_{K,\theta,c}(z) = \sqrt{\tau_{K,\theta,c}(H_{K,\theta,c}(z))}$.

Now, let $G_{K,\theta,c} = \log \circ \tau_{K,\theta,c}$.

This allows us to simplify the result of the previous slide to

$$G_{K,\theta,c}(H_{K,\theta,c}(z)) = 2G_{K,\theta,c}(z)$$

It immediately follows that

$$G_{K,\theta,c}(H^n_{K,\theta,c}(z)) = 2^n G_{K,\theta,c}(z)$$

Define $U_{K,\theta,c}(t) = \{z \in \mathbb{C} : G_{K,\theta,c}(z) > t\}$.

Since $\tau_{K,\theta,c}$ is continuous, $U_{K,\theta,c}(t)$ is open.

If $s < t$, then $U_{K,\theta,c}(t)$ is contained in the interior of $U_{K,\theta,c}(s)$.

For sufficiently large t, $U_{K,\theta,c}(t) \cup \{\infty\}$ is a simply connected subset of $\mathbb{C}_\infty$.

We have that $U_{K,\theta,c}(2^n t) = H^n_{K,\theta,c}(U_{K,\theta,c}(t))$.

The Riemann-Hurwitz formula applies to quasireguler mappings of degree d.
(Fletcher and Goodman)

The branch points of $H_{K,\theta,c}$ are 0 and ∞.

The set $\mathbb{C} \setminus U_{K,\theta,c}(t)$ consists of 2^m components, where m is some non-negative integer. It is connected iff $t \geq G_{K,\theta,c}(0)$.

Theorem: If $H^n_{K,\theta,c}$ has at least $2^n + 1$ fixed points, then $\partial I(H_{K,\theta,c})$ is not totally disconnected.

Let's get an idea of why this works.

If $0 \in I(H_{K,\theta,c})$, set $t_0 = G_{K,\theta,c}(0) > 0$.

Choose $t_1 \in (t_0, 2t_0)$.

Then $\mathbb{C} \setminus U_{K,\theta,c}(t_1)$ has one component, while $\mathbb{C} \setminus U_{K,\theta,c}(\frac{t_1}{2^n})$ has 2^n components.

Note that $U_{K,\theta,c}(\frac{t_1}{2^n}) = H_{K,\theta,c}^{-n}(U_{K,\theta,c}(t_1))$.

Note that $I(H_{K,\theta,c}) = \bigcup_{j=1}^{\infty} H_{K,\theta,c}^{-j}(U_{K,\theta,c}(t_1))$.

There are fixed points z_1 and z_2 in one of these components, which correspond to an inverse branch f of $H_{K,\theta,c}$ from $\mathbb{C} \setminus U_{K,\theta,c}(t_1)$ to $\mathbb{C} \setminus U_{K,\theta,c}(\frac{t_1}{2^n})$.

Both z_1 and z_2 must be stuck in $f^m(\mathbb{C} \setminus U_{K,\theta,c}(t_1))$ for all $m \in \mathbb{N}$.

The condition of the previous theorem can be met in cases where $c \notin \mathcal{M}_{K,\theta}$.

For example if $K > 4$ and $c \in (\frac{1}{2K} - \frac{1}{4}, -\frac{2}{K^2})$, then $H_{K,0,c}$ has 4 fixed points and $c \notin \mathcal{M}_{K,\theta}$.