Moduli Space of Flow Lines

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Two questions: Show $M(a,b)$ and $L(a,b)$ are orientable. Show $L(a,b)$ is diffeo to $M(a,b) \cap f^{-1}(\alpha)$.

Hutchings' notes shows a way to orient the Moduli Space. But we incur into issues, because it is not clear that we can translate the subspace $T_qD(q) \subset T_qX$ along γ while keeping it complementary to $TA(q)$. That is we may have that the x is converging to q, but that doesn't mean a chosen vector $v \in T_xX$ is converging to a vector in T_qX. **Update:** Yes it does, only because q is a nondegenerate critical point.

Proposition 15.21 says that if $M(a,b)$ is orientable manifold, and $L(a,b)$ is an immersed hypersurface, then if X is a vector field along $L(a,b)$ that is nowhere tangent to L(a,b), then $L(a,b)$ has a unique orientation such that for each p in $L(a,b)$, $(E_1,...,E_k)$ is an oriented basis for $T_p(L(a,b))$ iff $(X_p,E_1,...,E_k)$ is an oriented basis for $T_pM(a,b)$. If ω is an orientation form for M(a,b), then $\iota^*_{L(a,b)}(X \lrcorner \omega)$ is an orientation form for $L(a,b)$ with respect to this orientation, where $\iota : L(a,b) \hookrightarrow M(a,b)$ is the inclusion.

So to show the moduli is orientable we just need to show the $M(a,b)$ is orientable!

Exercise 14 (Audin Damian). Let E and F be two vector supspaces of a finite-dimensional real vector space. Show that an orientation of E is an equivalence class of bases of E for the equivalence relation

$$\mathcal{B} \sim \mathcal{B}' \iff \det_{\mathcal{B}} \mathcal{B}' > 0.$$

Likewise, verify that the relation

$$\mathcal{B} \sim \mathcal{B}' \iff \det_{(\mathcal{B},\mathcal{B}_0)} (\mathcal{B}', \mathcal{B}_0) > 0$$

defines an equivalence relation on the bases of the complements of F that does not depend on the chosen basis $\mathcal{B}_0$ of F. The equivalence classes are the co-orientations of F. Verify that if E is oriented, F is co-oriented and E and F are transversal, then $E \cap F$ is co-oriented.

Proof. An orientation of E is an equivalence class of bases of E for the equivalence relation B sim B' if and only if the determinant for the change of basis map is positive. In fact, every determinant is nonzero, and is thus positive or negative. The fact that det AB = det A . det B and det I > 0 shows that it does form an equivalence relation.

Now the other relation is on the bases of the complements of F. First we fix a basis B0, and then show that this equivalence relation does not depend on the chosen basis. That is, given two basis B and B' for the complement of F (so B and B0 form a basis of the ambient vector space), we say that B is equivalent to B' if the determinant of the change of basis (B, B0) to (B', B0) is positive. This is simply saying that given the orientation of the ambient vector space, we can deduce the equivalence class of B and B' based on whether (B, B0) and (B', B0) gives a positively oriented or negatively oriented basis. If both pairs give the same orientation, then the determinant of the change of basis should be positive. Now if (B, B0) to (B', B0) is positive, as well as (B', B0) to (B", B0), again by the product-respecting property of the determinant (B, B0) and (B', B0) have the same orientation.

Also note that the matrix of these changes of basis will be the identity on the last dim B0 minor. This shows the co-orientation of F is also equivalent to the orientation of the quotient vector space, complement of F.

Lastly, we just check that choosing different basis B1 for F does not matter. That is, we show that for instance, if B and B' in the complement of F are in the same equivalence class as determined with the basis B0, with B" in the other class, then B and B' will be in the same equivalence class as determined by the basis B1, and B" in the other.

By hypothesis (B, B0) to (B', B0) has positive determinant. Since the last dim B0 minor is the identity, then the first dim B minor is a matrix with positive determinant. This shows (B, B1) to (B', B1) has positive determinant as well, as we wanted.

By hypothesis (B, B0) to (B", B0) has negative determinant. This shows the first dim B minor has negative determinant, and as above (B, B1) to (B", B1) has negative determinant as well.

If E is oriented, and F is co-oriented, with E and F transversal, then at every point the tangent space of $E \cap F$ can be given a coherent orientation: If a basis $(E_1, ..., E_k)$ for $E \cap F$ is completed into a basis of E, say $(E_1, ..., E_k, E_{k+1}, ..., E_m)$, we can give an orientation for it. Then we complete it into a basis for the whole ambient space, say $(E_1, ..., E_k, E_{k+1}, ..., E_m, F_1, ..., F_l)$, which is automatically giving a basis for F via $(E_1, ..., E_k, F_1, ..., F_l)$, by transversality. Now we can't orient the basis for F, but we can orient the basis for the complement of F, namely $(E_{k+1}, ..., E_m)$. This then means that $E \cap F$ is co-oriented as a subspace of E. Since E is oriented as well, we induce an orientation on $(E_1, ..., E_k)$, as we wanted to show.

In the case of a manifold, the argument above is given in the language of local frames. At every point there is a continuous choice of frame for $E \cap F$ which patch together because the frames of E and the frames for the complement of F patch together already. $\square$

1 The moduli space is orientable

Both of $W^{(u)}(a)$ and $W^{(s)}(b)$ are orientable, so by setting once and for all orientations for $W^{(s)}(p)$ we co-orient $W^{(u)}(p)$ (at the critical points, which extend to the other points of it).

We may then use the exercise above to claim an orientation for the intersection manifolds $L(a, b)$.

Then again, we can use that each level set of the manifold V is co-oriented (by the pseudo-gradient vector field), that each $M(a, b)$ is oriented, and that $M(a, b)$ is transversal to $f^{-1}(\alpha)$, to define an orientation on the intersection, which is homeomorphic to $L(a, b)$. This shows each moduli space is orientable.

Succinctly,

$$\mathcal{T}W^s(a) = \mathcal{T}M(a, b) \oplus \mathcal{N}W^u(b)$$

$$\mathcal{T}M(a, b) = \mathcal{T}\mathcal{L}(a, b) \oplus \mathcal{N}f^{-1}(\alpha) = \mathcal{T}\mathcal{L}(a, b) \oplus \mathbb{R}$$

2 The moduli space (which has the quotient topology) is homeomorphic to the intersection of the trajectories with the level set, given the subspace topology

We have a clear bijection between $L(a,b)$ and $M(a,b) \cap f^{-1}(\alpha)$. It maps a flow line to the intersection with the level set. And the point is mapped to its integral curve. We show that both maps are continuous.

First, take an open set in $L(a,b)$. It has a pre-image which is open in $M(a,b)$, namely the points in its trajectories. This open set in $M(a,b)$ then intersected with the level set gives an open set in $M(a,b) \cap f^{-1}(\alpha)$, which also coincides with its pre-image under the map

$$M(a,b) \cap f^{-1}(\alpha) \to L(a,b).$$

This map is thus continuous.

Now take an open set in $M(a,b) \cap f^{-1}(\alpha)$, which we call C. It has a tubular neighborhood induced from the pseudo-gradient vector field, which we call B. Now this tubular neighborhood is open in V, and thus its intersection with $M(a,b)$ (which is itself, since the tubular neighborhood only has points in the integral curves of C) is open in $M(a,b)$.

Now we also have that the union of all points in trajectories of points in B, henceforth called A, is also open in $M(a,b)$, but furthermore saturated under the quotient map $M(a,b) \to L(a,b)$. To see that A is open, just take a point p in A. Then consider a translate $\varphi^t(p)$ of it which lies in B. Now take an open set U in $M(a,b)$ of this translate. Then translate it back, $\varphi^{-t}(U)$, which will also be open in $M(a,b)$, fully in A. This is because φ^{-t} is a diffeomorphism of V, which restricts to a diffeomorphism of $M(a,b)$ to itself.

This shows p is in the interior of A.

Now the image of this A under the quotient $M(a,b) \to L(a,b)$ is open, and it is clear that it is the pre-image of C under the map $L(a,b) \to M(a,b) \cap f^{-1}(\alpha)$. This shows this map is continuous.

3 Establishing orientations for the purpose of computing the Morse Homology with $\mathbb{Z}$ coefficients

To compute the Morse Homology in a manifold, all we need to do is to compute the signs of the trajectories between critical points a and b, where $Ind(a) = Ind(b)+1$. Then the chain groups will be each generated by the critical points of the respective indices, and the differential map will be

$$\partial^k(a) = \sum_{i=1}^{\#\mathrm{Crit}_f(k-1)} N(a,b_i)b_i,$$

where $b_1, ..., b_{\#\mathrm{Crit}_f(k-1)}$ are the critical points of index $k-1$, and a is a critical point of index k. Here $N(a,b_i)$ is the sum of the signs of the trajectories from a to b_i (which are finitely many).

We give a general algorithm to decide the orientations of the moduli spaces, but we use it mainly to establish the needed signs for the Morse Homology. More specifically, this algorithm generates a random frame for a chosen moduli space and determines whether it is posively or negatively oriented. This effectively determines the orientation in that moduli space by comparison with this one frame.

1) Choose an orientation for all stable manifolds.

2) Mark the co-orientation this induces in all unstable manifolds. That is, orient the normal bundle of each unstable manifold by the orientation already given at the critical point (that will be our positive orientation). Remember, at the critical point the normal vector space to the unstable manifold coincides with the tangent space to the stable manifold.

3) Given two critical points a and b with $Ind(a) > Ind(b)$, consider the intersection manifold $W^s(b) \cap W^u(a)$. Assume that $W^s(b) \cap W^u(a)$ is connected (if it is not we just take connected components and repeat the algorithm for each). Build a frame for this component of $W^s(b) \cap W^u(a)$ starting with the vector field $-\nabla f$. Then complete such frame into a positively oriented frame for $W^s(b)$ (that is, positively oriented according to the choice made in step 1).

4) In this process we indirectly built a normal frame for $W^u(a)$, namely by discarding the first $\dim(W^s(b) \cap W^u(a))$ vector fields from the frame of the previous step.

5) If this ordered normal frame is positively oriented according to the orientation chosen in step 2, the frame of $W^s(b) \cap W^u(a)$ built in step 3 is a positively oriented frame. If this ordered normal frame is negatively oriented according to the orientation chosen in step 2, the frame of $W^s(b) \cap W^u(a)$ built in step 3 is a negatively oriented frame.

Remark for Ind(a) - Ind(b) = 1: In the case where the indices differ by one, the frame built in step 3 will always be $[-\nabla f]$ itself. Thus, if the first option of step 5 holds, removing $-\nabla f$ we have a positively oriented empty frame for this connected component of the moduli space (a singleton consisting of one trajectory). This means this trajectory should have a positive sign. If the second option holds, we give a negative sign to the trajectory instead.

The reason why this algorithm works is because it effectively builds a frame for $W^s(b) \cap W^u(a)$ and determines whether it is positively or negatively oriented, (according to the induced orientation the choices of step 1 give).

On the one hand, the algorithm forces a positively oriented frame for $W^s(b)$. This means the ordered frame the algorithm generates for $W^s(b) \cap W^u(a)$ is positively oriented if and only if the extra vector fields used to complete it into a positive frame for $W^s(b)$ forms a positively oriented normal frame for $W^u(a)$. This is determined in steps 4 and 5.

On the other hand, the frame for $W^s(b) \cap W^u(a)$ we built is forced to have $-\nabla f$ as its first element. We are effectively granting that it will induce a positively oriented normal frame on the level sets (by discarding all but the first element).

This means that if the ordered frame we built for $W^s(b) \cap W^u(a)$ is positively oriented, removing the first element $-\nabla f$ gives a positively oriented frame on the moduli space $\mathcal{L}(a, b)$, (according to the orientation the choices of step 1 give to it).

Conversely, if we built a negatively oriented frame for $W^s(b) \cap W^u(a)$, then removing the first element $-\nabla f$ should give a negatively oriented frame for the moduli space.

4 Reference for a Proof that $\partial^2 = 0$

Can be found in `https://arxiv.org/pdf/math/0411465.pdf`, pages 21 to 26.

The main part of the proof is showing that the product of the induced signs in two trajectories $u \in \mathcal{L}(a,b), v \in \mathcal{L}(b,c)$ which together form one boundary point of $\mathcal{L}(a,c)$ is the same as the Stokes' orientation this boundary point (u,v) gets from the orientation of $\mathcal{L}(a,c)$.

Since the sum of the Stokes' orientations of the boundary points of an oriented compact 1-manifold is always zero, one gets $\partial^2 = 0$.

The main idea of the proof of this key statement above is that although b is not in any of $M(a,b), M(b,c), M(a,c)$, it is a limit point of the three. In this way, each manifold induces an orientation on the tangent space of b, but only because b is a nondegenerate critical point (If it were a regular point it would work as well, but the key here is that because of the Morse chart we know how the flow around the critical point behaves).

This allows us to work the relations on the tangent space of b, as follows:

$$\begin{aligned} T_bW^s(c) &= N_bW^u(b) \oplus T_bM(b,c) \\ &= T_bW^s(b) \oplus T_bM(b,c) \\ &= N_bW^u(a) \oplus T_bM(a,b) \oplus T_bM(b,c) \end{aligned} \tag{1}$$

(In particular, notice that putting the normal bundle first then the moduli space is an arbitrary convention in determining the induced orientations, but as long as it is done consistently any choice will give the same result, namely that the differential squared is zero).

on the one hand, and on the other since b is a limit point of $M(a,c)$,

$$T_bW^s(c) = NW^u(a) \oplus T_bM(a,c),$$

which implies that $T_bM(a,b) \oplus T_bM(b,c)$ and $T_bM(a,c)$ have the same orientation.

In particular, the orientation in the trajectory u of $T_bM(a,b)$ can be written $n_u[\dot{u}(+\infty)]$ and v of $T_bM(b,c)$ as $n_v[\dot{v}(-\infty)]$.

The orientation $n_un_v[\dot{u}(+\infty)] \oplus [\dot{v}(-\infty)]$ on $T_bM(a,c)$ corresponds to the orientation

$$n_un_v[-\nabla f, \frac{d}{dt}\psi],$$

on $TM(a,c)$, as one can see from the Morse chart. In this orientation above $\psi : [0,+\infty) \to M(a,c) \cap f^{-1}(\alpha)$ is the boundary chart that $\mathcal{L}(a,c)$ admits (cf. p.61-62, Prop. 3.2.8 on Audin-Damian).

This means that $n_un_v[\dot{u}(+\infty)] \oplus [\dot{v}(-\infty)]$, the positive orientation of $TM(a,b) \oplus TM(b,c)$ is the same as the positive orientation $n_un_v[-\nabla f, \frac{d}{dt}\psi]$ of $TM(a,c)$. Moreover, this orientation induces the positive orientation $-n_un_v[\frac{d}{dt}\psi]$ of $\mathcal{L}(a,c)$ (again following a set convention of our choosing, namely that the induced orientations on the moduli spaces are done by removing the positively oriented normal bundle $-\nabla f$ from the end of any positively ordered basis).

Now since $-n_un_v[\frac{d}{dt}\psi]$ is the positive orientation of $\mathcal{L}(a,c)$, we compute the Stokes' orientation it induces on (u,v) the boundary point. It is n_un_v since $[\frac{d}{dt}\psi]$ induces a negative sign as it is leaving the broken trajectory (u,v) behind.

This procedure (following the fixed conventions of how orientations/coorientations of stable/unstable manifolds induce orientations on the moduli spaces) of comparing signs can be done at any broken trajectory.

We conclude that the Stokes' orientation on (u,v) corresponds to the product of the signs induced on u and v, which is what we needed to show for $\partial^2 = 0$.