Obstruction-theoretic definition of characteristic classes

1. **Stiefel manifolds and their low-dimensional homotopy groups**

The Stiefel manifold V_kR^n is the space of orthonormal k-frames at the origin in R^n. Such a frame can be represented as an $n \times k$ matrix, writing its component vectors as the k columns; two frames will be considered close if all the components of the corresponding matrices are close, i.e. we give V_kR^n the topology of a subset of the space of $n \times k$ matrices $= R^{kn}$.

In particular $V_1R^n = S^{n-1}$ the unit $(n-1)$-sphere, and $V_nR^n = O_n$ the orthogonal group.

The low-dimensional homotopy groups $\pi_iV_kR^n$, those for $i \le n-k$, can be conveniently calculated using the fibration $V_kR^{n+1} \to S^n$, which maps a frame to its last vector. The fiber is $V_{k-1}R^n$, and the inclusion of the fiber in the total space can be understood in terms of matrices as the map which takes a $(k-1) \times n$ matrix A to the $k \times (n+1)$ matrix

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

This fibration has exact homotopy sequence

$$\cdots \to \pi_iS^n \xrightarrow{\delta_*} \pi_{i-1}V_{k-1}R^n \to \pi_{i-1}V_kR^{n+1} \to \pi_{i-1}S^n \to \cdots.$$

When $k = 2$, the fibration $V_2R^{n+1} \to S^n$ is the unit tangent circle bundle to S^n, since the the second vector in the frame, being normal to the first one, can be considered a unit tangent vector at the corresponding point of S^n. The fiber is thus S^{n-1}.

In this case there is a useful geometric interpretation of the connecting homomorphism $\delta_*\colon \pi_nS^n \to \pi_{n-1}S^{n-1}$.

Consider S^n as the unit sphere in R^{n+1} and label $\mathbf{S} = (0, \cdots, -1)$ and $\mathbf{N} = (0, \cdots, 1)$ its south and north poles. The open set $S^n - \mathbf{S} - \mathbf{N}$ is homeomorphic to $S^{n-1} \times (-1, 1)$ by the map taking $(x_1, \cdots, x_n, x_{n+1})$ to the pair $((x_1/m, \cdots, x_n/m), x_{n+1})$ where $m = \sqrt{\sum_1^n x_i^2}$. The inverse of this homeomorphism may be extended continuously to $\phi\colon S^{n-1} \times [-1, 1] \to S^n$ by mapping $S^{n-1} \times \{-1\}$ to $\mathbf{S}$ and $S^{n-1} \times \{1\}$ to $\mathbf{N}$. In this way any map of S^n

can be considered as a homotopy of maps of S^{n-1}, parametrized by $[-1, 1]$, starting and ending with constant maps (Fig. 1).

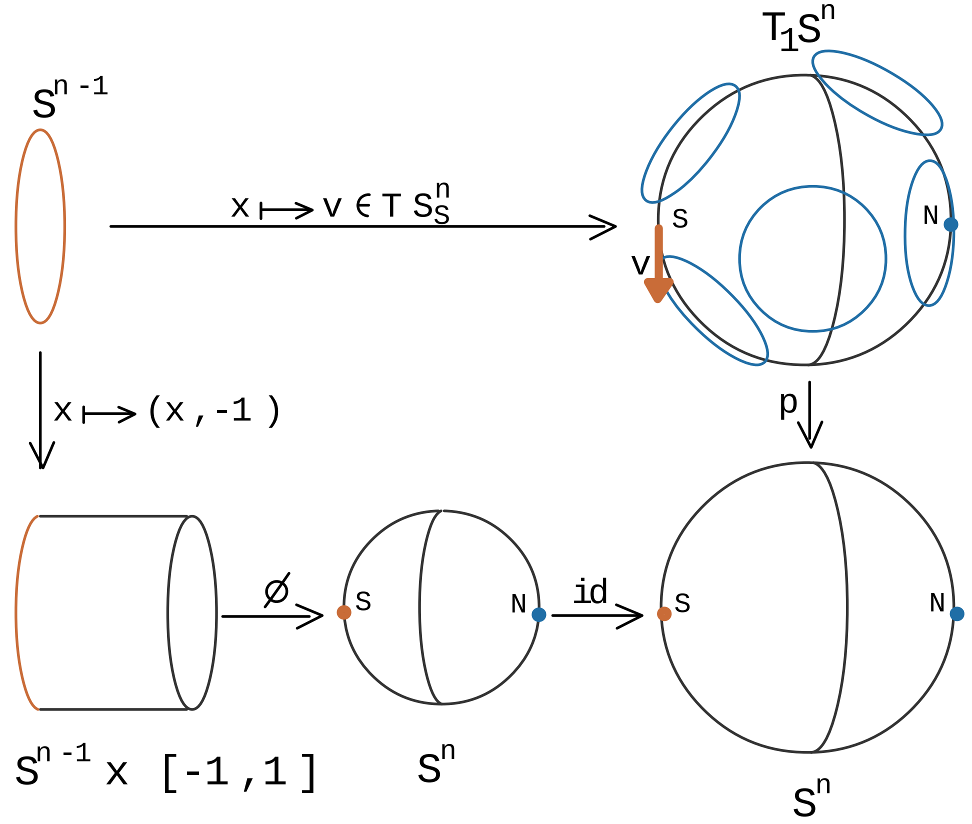

Figure 1:

This is true in particular of the identity map $id\colon S^n \to S^n$, which represents the generator, which we may call $[id]$, of $\pi_n S^n$. Suppose v is a unit tangent vector at the south pole. Then the constant map $S^{n-1} \mapsto v$ covers $id \circ \phi\colon S^{n-1} \times \{-1\} \mapsto \mathbf{S}$. The covering homotopy property, applied to the locally trivial fibration $T_1S^n \to S^n$ tells us that the homotopy $id \circ \phi$ can be lifted to $\Phi\colon S^{n-1} \times [-1,1] \to T_1S^n$ (Fig. 2).

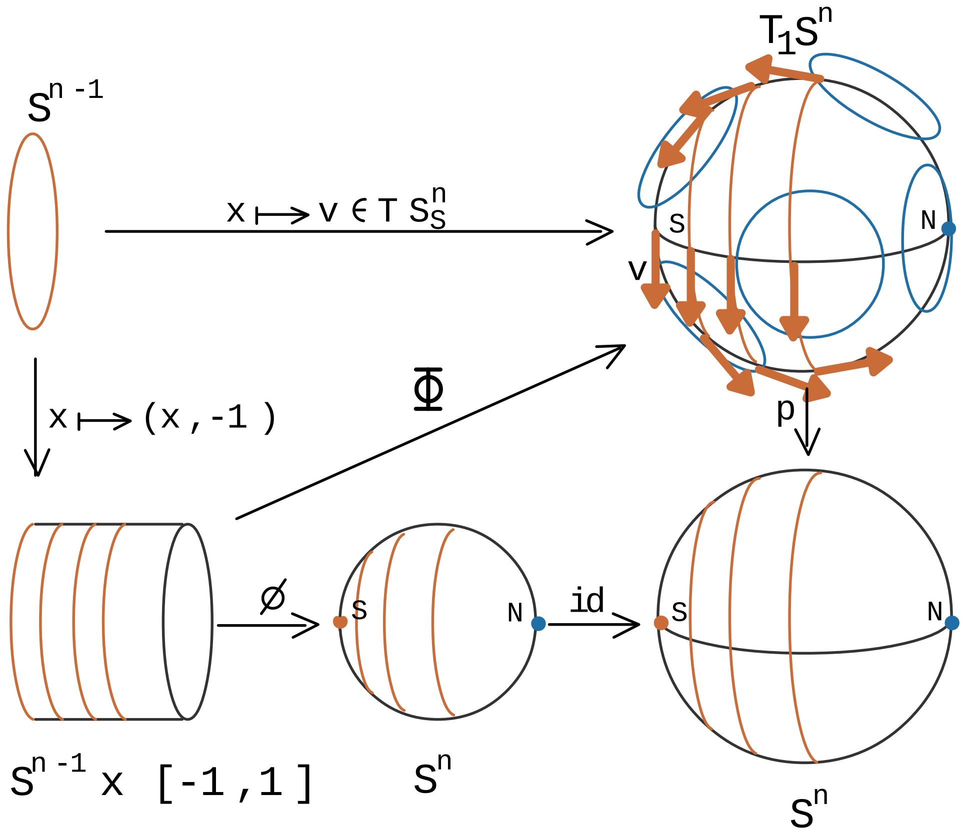

Figure 2:

In particular, Φ maps $S^{n-1} \times \{1\}$ to the fiber over $\mathbf{N}$, another copy of S^{n-1}. The homotopy class of this map is by definition $\delta_*([id]) \in \pi_{n-1}S^{n-1}$ (Fig. 3).

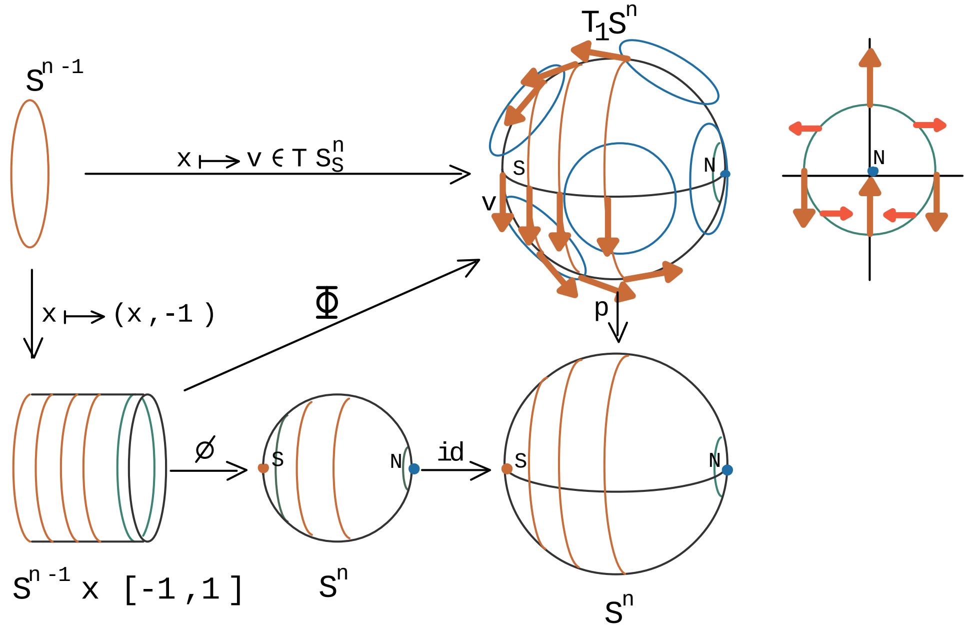

Figure 3:

Now $\Phi|S^{n-1} \times [-1,1)$ defines a unit vector field on $S^n - \mathbf{N}$. The number $\delta_*([id]) \in \pi_{n-1}S^{n-1} = \mathbf{Z}$ is therefore the Euler characteristic of S^n, i.e. 0 if n is odd, 2 if n is even. Summarizing this discussion in our original notation:

Proposition 1. In the exact homotopy sequence of the fibration $V_2R^{n+1} \to S^n$, with fiber $V_1R^n = S^{n-1}$ the connecting homomorphism $\delta_*\colon \pi_nS^n = \mathbf{Z} \to \pi_{n-1}V_1R^n = \mathbf{Z}$ takes 1 to 0 when n is odd, and to 2 when n is even.

Corollary 2. (a) $\pi_iV_2R^{n+1} = 0$ for $i \leq n-2$.

(b) $\pi_{n-1}V_2R^{n+1} = \begin{cases} \mathbf{Z} & (n \text{ odd}) \\ \mathbf{Z}_2 & (n \text{ even}) \end{cases}$.

Proof. (a) (Not really a corollary, but placed here for convenience). For

$i \leq n-2$, both $\pi_i S^n$ and $\pi_i V_1 R^n = \pi_i S^{n-1}$ are 0, so the segment

$$\pi_i V_1 R^n \to \pi_i V_2 R^{n+1} \to \pi_i S^n$$

from the exact homotopy sequence of our fibration becomes

$$0 \to \pi_i V_2 R^{n+1} \to 0.$$

(b) The exact sequence segment

$$\pi_n S^n \xrightarrow{\delta_*} \pi_{n-1} V_1 R^n \to \pi_{n-1} V_2 R^{n+1} \to \pi_{n-1} S^n \to$$

is either

$$\mathbf{Z} \xrightarrow{0} \mathbf{Z} \to \pi_{n-1} V_2 R^{n+1} \to 0 \quad (n \text{ odd})$$

or

$$\mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \pi_{n-1} V_2 R^{n+1} \to 0 \quad (n \text{ even}).$$

Lemma 3. For (*) $i \leq m-2$, the inclusion $V_j R^m \to V_{j+1} R^{m+1}$ induces an isomorphism $\pi_i V_j R^m \to \pi_i V_{j+1} R^{m+1}$.

Proof. Consider the exact homotopy sequence of the fibration $V_{j+1} R^{m+1} \to S^m$ with fiber $V_j R^m$:

$$\to \pi_{i+1} S^m \to \pi_i V_j R^m \to \pi_i V_{j+1} R^{m+1} \to \pi_i S^m \to .$$

Our hypothesis $i \leq m-2$ guarantees that the central map is an isomorphism, since then both $\pi_{i+1} S^m$ and $\pi_i S^m$ are 0.

Proposition 4
(a) $\pi_i V_k R^n = 0$ for $i \leq n-k-1$.
(b) $\pi_{n-1} V_1 R^n = \pi_{n-1} S^{n-1} = \mathbf{Z}$.
(c) For $k \geq 2$, $\pi_{n-k} V_k R^n = \begin{cases} \mathbf{Z} & (n-k \text{ even}) \\ \mathbf{Z_2} & (n-k \text{ odd}) \end{cases}$.

Proof. (a) If $i \leq n-k-1$, then $\pi_i S^{n-k} = 0$. Writing S^{n-k} as $V_1 R^{n-k+1}$, and applying Lemma 3, we obtain a sequence of isomorphisms induced by inclusions

$$0 = \pi_i V_1 R^{n-k+1} \simeq \pi_i V_2 R^{n-k+2} \simeq \cdots \simeq \pi_i V_{k-1} R^{n-1} \simeq \pi_i V_k R^n.$$

(b) This is clear since a 1-frame is a unit vector.

(c) By Corollary 2,

$$\pi_{n-k} V_2 R^{n-k+2} = \begin{cases} \mathbf{Z} & (n-k \text{ even}) \\ \mathbf{Z}_2 & (n-k \text{ odd}) \end{cases} .$$

Lemma 3 again gives a sequence of isomorphisms:

$$\pi_{n-k} V_2 R^{n-k+2} \simeq \pi_{n-k} V_3 R^{n-k+3} \simeq \cdots \simeq \pi_{n-k} V_{k-1} R^{n-1} \simeq \pi_{n-k} V_k R^n .$$

Note that in parts (a) and (c) the difference between the R exponent and the π index starts at 2 and increases, so hypothesis (*) of Lemma 3 is always satisfied.

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