A NOTE ON THE HISTORY OF TRIGONOMETRIC FUNCTIONS AND SUBSTITUTIONS

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Abstract: Trigonometric functions appear very frequently in mechanism kinematic equations (for example as soon a revolute joint is involved in the mechanism). Dealing with these functions is difficult and trigonometric substitutions are used to transform them into algebraic terms that can be handled more easily. We present briefly the origin of the trigonometric functions and of these substitutions.

Keywords: trigonometric substitutions, robotics

1 INTRODUCTION

When dealing with the kinematic equations of a mechanism involving revolute joints appears frequently terms involving the sine and cosine of the revolute joint angles. Hence when solving either the direct or inverse kinematic equations one has to deal with such terms. But the problem is that the geometric structure of the manifold build on such terms is not known and that only numerical method can be used to solve such equations. A classical approach to deal with this problem is to transform these terms into algebraic terms using the following substitution:

$$t = \tan(\frac{\theta}{2}) \Rightarrow \quad \sin(\theta) = \frac{2t}{1+t^2} \quad \cos(\theta) = \frac{1-t^2}{1+t^2}$$

This substitution allows one to transform an equation involving the sine and cosine of an angle into an algebraic equation. Such type of equation has a structure that has been well studied in the past and for which a large number properties has been established, such as the topology of the resulting manifold and its geometrical properties. Furthermore efficient formal/numerical methods are available for solving systems of algebraic equations.

But when asking in the community about the possible origin of this substitution I get only vague answer, most of them mentioning ancient Greek mathematicians. Hence the purpose of this paper is to investigate the origin of these substitutions.

2 THE ORIGIN OF THE TRIGONOMETRIC FUNCTIONS SINUS AND COSINUS

2.1 Origin for sinus and cosinus

The word "trigonometry" appears for the first time in the book *Trigonometria: sive de solutione triangulorum tractatus brevis et perspicuus* published by B. Pitiscus (1561-1613) in 1595 and means "the study of trigons" in Latin (trigon being the word used for triangle) with a first appearance in English in the 1614 translation of the book of Pitiscus by Ra. Handson [1, 3, 4]. But *trigonometric functions* (a term that according to Cajori was introduced in 1770 by Georg Simon Klügel (1739-1812) while the term *trigonometric equations* can be found in 1855 as a chapter title of the book "A treatise on plane and spherical trigonometry" published by William Chauvenet) were in use well before: Hipparchus of Rhodes (190-120 BC), also called Hipparchus of Bithynia, called the founder of trigonometry, publishes a lost work on the chords of a circle in 12 books (although this number is contested) in 140 BC [8] (the chord function Crd is related to the sine by $\sin(a) = Crd(2a)/120$), probably using Pythagora's theorem and the half-angle theorem. Despite the fact that Hipparchus was a major mathematician and astronomer it remains only one known of it writings, the *Commentary on Aratus and Eudoxus*, a minor work. One of the use of these tables was to tell the time of day or period of the year according charts of the angular measure between various stars compiled by astronomers.

Around AD 100 Menelaus (circa 70-130 AD) has published six lost book of tables of chords. In the two first books of its 13-books *Almagest* Ptolemy (85-125 AD) also gives a table of chords (note that Almagest is not the real name of the work of Ptolemy: originally the Greek title was *The Mathematical Compilation* that was soon replaced by the *Greatest Compilation* which was translated in Arabic as *Al-majisti* from which *Almagest* is derived).

The first appearance of the sine of an angle appears in the work of the Hindu Aryabhata the Elder (476-550), in about 500, that gives tables of half chords (that are 120 times the sine) based on the Greek half-angle formula and uses the word of *jya* to describe these quantities [5, 6, 9]. The same sort of table was presented by Brahmagupta (in 628) and a detailed method for constructing table of sine was presented by Bhaskara in 1150.

The Hindu word *jya* was phonetically reproduced by the Arabs as *jiba*, a word that has initially no meaning. But *jiba* became *jaib* in later Arab writings, a word that has the meaning of "fold". When Europeans translated the Arabic mathematical works into Latin, they translated *jaib* into *sinus* meaning fold, bay or inlet in Latin: especially Fibonnacci's use of the term *sinus rectus arcus* was one of the main step for the universal use of the word *sinus*. Note that the first appearance of *sinus* is still a subject of controversy:

- according to Cajori (1906) *sinus* appears in the 1116 translation of the astronomy of Al Battani (that formally introduces the cosine) by Plato of Tivoli that was published in 1537

- Eves claimed that *sinus* appears in the translation of the Algebra (*al-jabr w'al-muqabala*, the science of transposition and cancellation) of Al-khowarizmi (circa 780-850), a mathematician, astronomer and geographer, by Cremona (1114-1187)

- Boyer claims that *sinus* appears in 1145 in the translation of the tables of Al-khowarizmi provided in *Sindhind zij* by Robert of Chester (or Robert from Ketton, ?) (his translation of the treatise on algebra starts with *Dixit Algorithmi: laudes deo rectori nostro atque defensori dicmus dignas*, "Algorithmi says: praise be to God, our Lord and Defender", and this was the first occurence of a sentence that will lead to the modern word of algorithm). Robert of Chester was a translator that was hired by the Castillian (Spanish) king Alphonso the 6th who captured Toledo from the Arabs and found a large library with many Arab manuscripts, including translations of Greek books unknown in the rest of Europe.

Georg Joachim von Lauchen Rheticus (1514-1574) published in 1542 some chapters of Copernicus's book giving all the trigonometry relevant to astronomy and produced accurate tables of the 6 trigonometric functions that were published after his death in *Opus Palatinum de triangulis* or *Canon of the Doctrine of Triangles* (probably written in 1551 but published in 1596).

Johann Müller of Königsberg also called Regiomontanus (1436-1476) writes the book *De triangulis omnimodis* that includes accurate data on the sine and its inverse that were done around 1464 but published only in 1533.

The word *cosinus* has a similar development: Viète (1540-1603) uses the term *sinus residae* while Edmund Gunter (1581-1626), a Rector and professor of astronomy, suggested the word *co-sinus* in 1620.

2.2 The sine and cosine abbreviation

In 1624 Edmund Gunter uses the abbreviation *sin* in a drawing (but the term was not used in the text) while it is claimed that it appears for the first time in the book *Cursus mathematicus* of the French mathematician Pierre Hérigone (1580-1643) published in 1634: this claim is controversial as many works relates that none of the notation proposed by Hérigone was used afterwards except for the $\angle$ one. Some other authors claimed that William Oughtred (1575-1660), the English rector of Albury, uses also *sin* in its book *Addition vnto the Vse of the Instrvment called the Circles of Proportion* published in 1632.

Other abbreviation were used: *Si* by Cavalieri (1598-1647), an Italian Jesuit mathematician, *S* by Oughtred (1574-1660), an English Episcopal minister who became passionately interested in mathematics, in its book *Trigonometrie* published in 1657. The term *sin.* (with a period) was proposed by Thomas Fincke (1561-1656), a Danish professor in rethorics, medicine and mathematics, in 1583 in his book *Geometriae rotundi.*

As for the cosine Cavalieri was using the notation *Si.2*, Oughtred using *s co arc* or *sco* and Sir Jonas Moore (1627-1679) proposes *Cos.* in *Mathematical Compendium* (1674). John Wallis (1616-1703) was using S while Samuel Jeake (1623-1690) used *cos.* in Arithmetick published in 1696.

The earliest use of *cos* is attributed either to Euler in 1729 in *Commentarii Academiae Scient. Petropollitanae, ad annum 1729* or to William Oughtred either in 1631 or 1657.

The modern presentation of trigonometry can be attributed to Euler (1707-1783) who presented in *Introductio in analysin infinitorum* (1748) the sine and cosine as functions rather than as chords.

2.3 Trigonometric substitution in sine and cosine

Ptolemy was aware of the formula

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

and in 980 the Arab Abu'l-Wafa was using the formula

$$\sin 2x = 2 \sin x \cos x$$

in its book *Kitab al-Khamil*. This substitution was essential to calculate the tables that were used for astronomy and engineering.

Now consider the substitution:

$$\cos s \cos t = \frac{\cos(s + t) + \cos(s - t)}{2}$$

Tycho Brahe (1546-1601) uses this substitution to perform the multiplication of 2 number using an algorithm known as *prosthaphaeresis*. Assume that you want to multiply x times y. You first look a cosine table to look up the angle s whose cosine is x and the angle t whose cosine is y and then determines what are the cosines of $s + t$ and of $s - t$. If you average these two cosines you get the product xy.

3 THE ORIGIN OF THE TRIGONOMETRIC FUNCTION TANGENT

Tangent was initially not associated to angles or circles but to the length of the shadow that is projected by an object and that was used, for example, by Thales to measure the height of the pyramids.

The first known shadows tables were produced by the Arabs around 860 using both the tangent and the cotangent that were translated into Latin as *umbra recta* and *umbra versa*.

The first appearance of the term tangens (from the Latin *tangere*, to touch) is due to Thomas Fincke (1561-1656) used in 1583 in its Book 14 of *Geometrica rotundi* and was also used in 1632 by William Oughtred in *The circles of*

Proportion. Viète was using the terms *amsinus* and *prosinus* and *sinus foecundarum* because he did not approve of the term tangent because it could be confused with the term in geometry..

The term cotangens was used by Edmund Gunter in 1620, *cot.* by Samuel Jeake in 1696. Finally *cot* was proposed by A.G. Kästner in *Anfangsgründe der Arithmetik.*

As for the notation Cavalieri was using *Ta* and *Ta.2*, Oughtred *t arc* and *t co arc* and Wallis *T* and *t*. The modern notation *tan* appears in a book of Albert Girard (1595-1632), a French musician settled in the Netherlands with an interest in algebra and military engineering, in 1626 and in the drawings of Edmund Gunter but was written as

$$\begin{array}{c}\tan\\ \mathrm{A}\end{array}$$

The notation *cot* was proposed by Sir Jonas Moore in 1674.

4 THE HALF-ANGLE TANGENT SUBSTITUTION

As seen previously the half-angle sine formula was used very early. This is not the case of the transformation:

$$t = \tan(\frac{\theta}{2}) \Rightarrow \quad \sin(\theta) = \frac{2t}{1+t^2} \quad \cos(\theta) = \frac{1-t^2}{1+t^2} \tag{1}$$

All the authors seem to agree that this substitution was first used by Weierstrass (1815-1897) and is often called *Weierstrass substitution* of *Weierstrass t-substitution* [7]. Weierstrass was interested in the integration of rational functions of $\sin(\theta)$ and $\cos(\theta)$. In addition to equation (1) it is indeed easy to prove that

$$d\theta = \frac{2dt}{1+t^2} \tag{2}$$

Combining equations (1) and (2) any integrand containing a rational function of $\sin(\theta)$ and $\cos(\theta)$ can be converted to an integrand containing a rational function of t. When considering integrals this substitution may lead to spurious discontinuities [2]. For example the function $3/(5-4\cos x)$ is continuous and positive for all real x, and so its integral should be continuous and monotically increasing. Using the Weierstrass substitution we get

$$\int \frac{3dx}{5-4\cos x} = \int \frac{6du}{1+9u^2} = 2arctan(3\tan(x/2))$$

which is discontinuous at odd multiples of π.

5 CONCLUSION

Although trigonometric functions and substitutions are widely used in the mechanism community their history is not so much known. The purpose of this note was to emphasize some key points in their discovery. An open question remains: who is the first researcher that uses the Weierstrass substitution for the purpose of the kinematic analysis of a mechanism ?

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- MacTutor History of Mathematics archive:
  `http://turnbull.mcs.st-and.ac.uk/~history/`
- geometry:
  `http://www.geometry.net/index.html`
- History of Mathematics:
  `http://aleph0.clarku.edu/~djoyce/mathhist/`
- Math Archives:
  `http://archives.math.utk.edu/topics/trigonometry.html`
- Earliest Known Uses of Some of the Words of Mathematics :
  `http://members.aol.com/jeff570/mathword.html`
- The Galileo Project:
  `http://es.rice.edu/ES/humsoc/Galileo/`

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