Trigonometry for Adults

Tony Phillips, November 2014

Students need help in making the transition from high-school trigonometry (based on right-triangle trig, an awkward conversion to the circle – "quadrants" – and measurement mostly in degrees) to the trigonometry used in college and beyond (emphasis on sin, cos, etc. as *functions* of a variable measured in radians). Here's what we can do to make it easier for them.

I. Whenever a trigonometry problem arises, put this diagram on the board. Encourage your students to copy this diagram and to learn to reproduce it themselves.

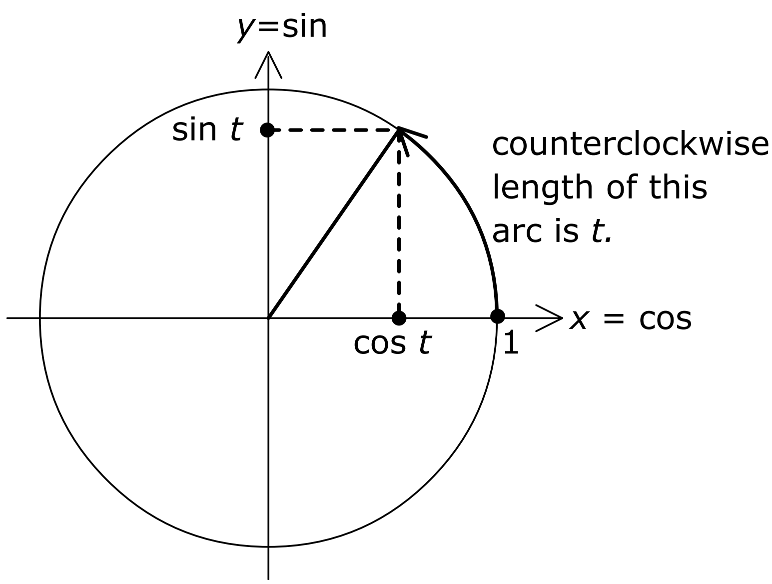

Figure 1: The "trigonometric circle." When t is measured counterclockwise along the unit circle starting from 1 on the horizontal axis, $\cos t$ and $\sin t$ are the x and y coordinates of t.

If you and your students work from the trigonometric circle, you can't go wrong. Many important identities, e.g. $\sin^2 t + \cos^2 t = 1$, $\quad \sin(t+\pi) = -\sin t$, $\quad \cos(t+\pi) = -\cos t$ can be deduced or reliably checked using this diagram.

II. Radian measure and degrees.

The trigonometric circle can also help clarify the relation between degree-measure and radian-measure of an angle.

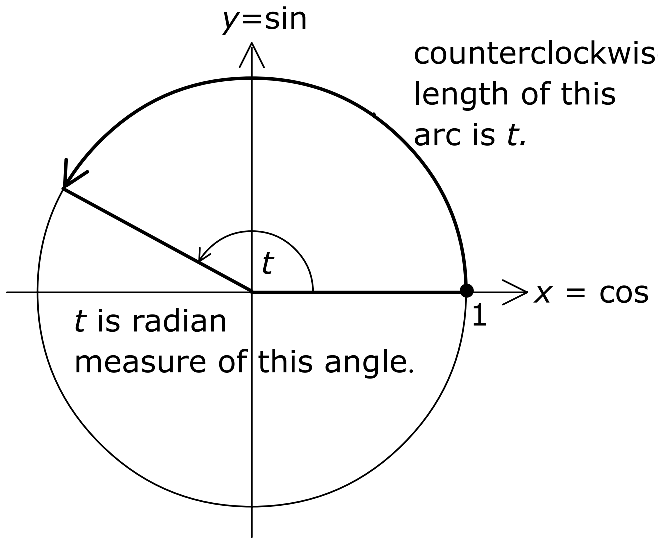

Figure 2: The radian measure of an angle is determined by drawing it at the center of the trigonometric circle, with the leading edge along the positive x-axis, and reading off the length of the intercepted arc.

You can explain to your students that degree-measurement of angles is a cultural artifact dating back to the Old Babylonians (c. 2000 BC), who used a base-60 representation of numbers. Radian measure, geometrically more intrinsic, is fundamental to the use of trigonometric functions in calculus. For example, the approximation $\sin t \approx t$, which works very well when $|t|$ is small, only works when t is measured in radians.

The trigonometric circle gives a reliable method for converting degrees to radians, and for estimating how big an angle is from its radian measure. For example, since $60°$ is one-sixth of the "total angle" $360°$, the corresponding radian measure will be one-sixth of the complete circumference of the unit circle, i.e. $2\pi/6 = \pi/3$. On the other hand, since $\pi/2$ is 1.570..., radian measure 1, which is about 2/3 of that, corresponds to about $(2/3) \times 90° = 60°$.

III. Extend the use of the trigonometric circle to tan.

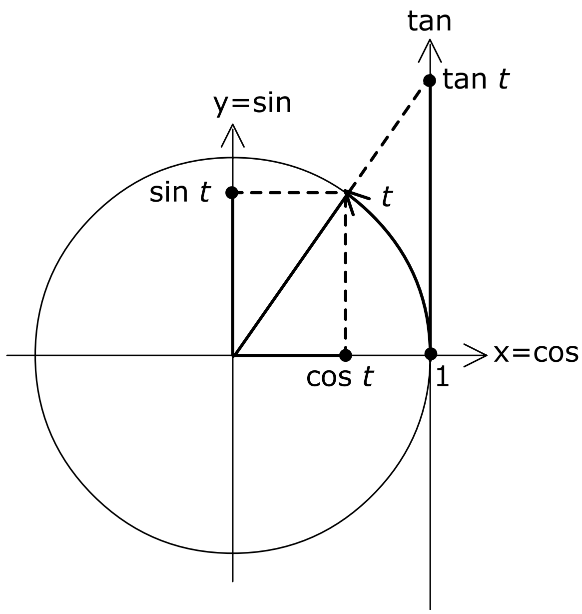

Figure 3: The trigonometric circle gives a reliable definition of the tangent function. Extend the radius line from the center to the end of the arc t (in either direction) until it intersects the "tangents" axis: the oriented line tangent to the circle at $x = 1$. The height of that intersection point is $\tan t$. [There are two congruent right triangles in the figure, one with legs $\cos t$ and $\sin t$, the other with corresponding legs 1 and $\sin t/\cos t = \tan t$].

This representation allows reliable retrieval of facts about the function $\tan t$. For example, $\lim_{t\to\pi/2} \tan t = +\infty$, $\lim_{t\to-\pi/2} \tan t = -\infty$, $\tan(t + \pi) = \tan t$.

IV. Connecting with right-triangle trigonometry.

It is essential to maintain the link to the right-triangle-trigonometry definitions of $\sin t, \cos t, \tan t$ when t is one of the acute angles in a right triangle. Often in calculus right-angle-trigonometry is used to set up the problem, and then knowledge of the behavior of sin, cos, tan as functions is used to solve it.

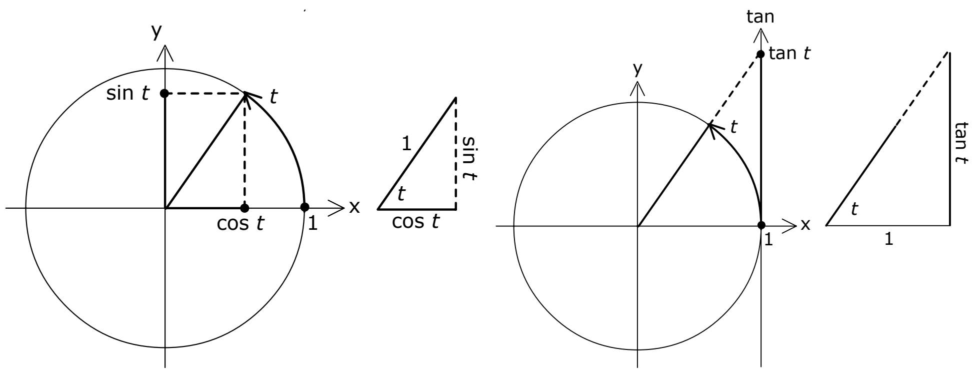

Figure 4: When t is the radian measure of one of the acute angles of a right triangle, then $\sin t$ is the ratio of the lengths: (opposite side)/(hypothenuse), $\cos t$ is the ratio of the lengths: (adjacent side)/(hypothenuse), and $\tan t$ is the ratio of the lengths: (opposite side)/(adjacent side).

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In particular the values of the trigonometric functions at the canonical angles $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ can be rapidly and reliably derived from elementary geometry and Pythagoras' Theorem. You and all your students should know how to recalculate these values when necessary.

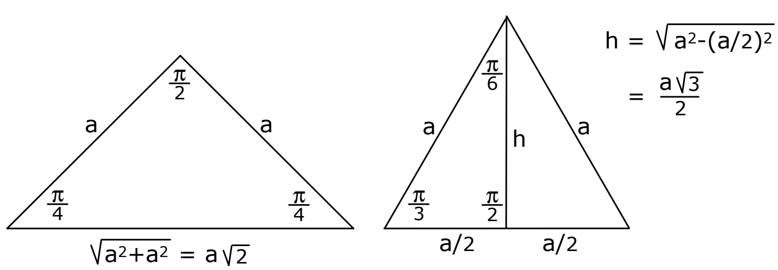

Figure 5: *Right isosceles triangle:* $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ also, giving $\tan\frac{\pi}{4} = 1$. *Equilateral triangle:* $h = \sqrt{a^2 - (a/2)^2} = \sqrt{3a^2/4} = \frac{a\sqrt{3}}{2}$. So $\sin\frac{\pi}{3} = \cos\frac{\pi}{4}6 = \frac{\sqrt{3}}{2}$, and $\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}$. This gives $\tan\frac{\pi}{3} = \sqrt{3}$ and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

V. Graphing $\sin t$.

When sketching the graph of $\sin t$ (or $\cos t$) try to remember that π is a little more than 3, whereas $|\sin t| \leq 1$, and to scale your graph realistically. This will help your students grasp and remember the geometric implications of $\sin'(0) = \cos(0) = 1$ and $\sin'(\pi) = \cos(\pi) = -1$, as well as $\int_0^\pi \sin t \; dt = 2$, etc.

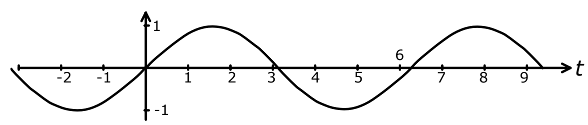

Figure 6: A sketch of the graph of $\sin t$ trying to respect the scale.