Midterm 1 Solutions

Note that there different forms of this test; yours may be slightly different from this one.

1. (a) (15 points) What are the 4 fourth roots of -9?

   Write -9 as $9e^{i\pi}$. Then the rule for n-th roots gives the four roots as

   $$\sqrt[4]{9}e^{i(\frac{\pi}{4}+\frac{2k\pi}{4})}, \quad k = 0, 1, 2, 3.$$

   (b) (15 points) Write $z^4 + 9$ as $(z - r_1)(z - r_2)(z - r_3)(z - r_4)$.

   The four roots calculated above are:

   $$\begin{array}{rcl} r_1 = \sqrt{3}e^{i\frac{\pi}{4}} & = & \sqrt{3}\frac{1+i}{\sqrt{2}} \\ r_2 = \sqrt{3}e^{i(\frac{\pi}{4}+\frac{2\pi}{4})} & = & \sqrt{3}\frac{-1+i}{\sqrt{2}} \\ r_3 = \sqrt{3}e^{i(\frac{\pi}{4}+\frac{4\pi}{4})} & = & \sqrt{3}\frac{-1-i}{\sqrt{2}} \\ r_4 = \sqrt{3}e^{i(\frac{\pi}{4}+\frac{6\pi}{4})} & = & \sqrt{3}\frac{1-i}{\sqrt{2}} \end{array}$$

   This gives

   $$z^4+9 = (z-\sqrt{3}\frac{1+i}{\sqrt{2}})(z-\sqrt{3}\frac{-1+i}{\sqrt{2}})(z-\sqrt{3}\frac{-1-i}{\sqrt{2}})(z-\sqrt{3}\frac{1-i}{\sqrt{2}}).$$

   (c) (15 points) Use the fact that the complex roots of a polynomial with real coefficients come in *complex conjugate pairs* to write $z^4 + 9$ as a product of two quadratic polynomials with real coefficients.

   In this case r_1 and r_4 are complex conjugates, as are r_2 and r_3. To shorten notation, notice that $(z - a)(z - \overline{a}) = z^2 - 2\Re(a) + |a|^2$, where $\Re(a)$ is the real part of a. So:

   $$(z - r_1)(z - r_4) = z^2 - 2\sqrt{3}\frac{1}{\sqrt{2}}z + 3 = z^2 - \sqrt{6}z + 3$$

   and

   $$(z - r_2)(z - r_3) = z^2 - 2\sqrt{3}\frac{-1}{\sqrt{2}}z + 3 = z^2 + \sqrt{6}z + 3$$

so finally

$$z^4+9=(z^2-\sqrt{6}z+3)(z^2+\sqrt{6}z+3).$$

An alternative method was to write

$$z^4+9=(z^2+3i)(z^2-3i)$$

and

$$z^2+3i=(z-i\sqrt{3i})(z+i\sqrt{3i});\quad z^2-3i=(z+\sqrt{3i})(z-\sqrt{3i})$$

to get

$$z^4+9=(z-i\sqrt{3i})(z+i\sqrt{3i})(z+\sqrt{3i})(z-\sqrt{3i})$$

Here it's not obvious which roots are complex conjugates: best to work it out with $\sqrt{3i}=\sqrt{3}\frac{1+i}{\sqrt{2}}$.

2. (a) (15 points) What is the image of the line $\Im(z)=1$ [i.e. $\{x+iy|y=1\}$] under the mapping $w=z^2$?

The mapping $w=z^2$ takes (x,y) to $(u=x^2-y^2, v=2xy)$. So the line $\Im(z)=1$ goes to $(u=x^2-1, v=2x)$. [This is also true of the line $\Im(z)=-1$ given on some of the forms of the test]. The image of the line is therefore the parabola $(u=x^2-1, v=2x)$, or $u=(\frac{v}{2})^2-1$.

(b) (15 points) Sketch the image of the half-plane $\Im(z)\geq 1$ under the mapping $w=z^2$.

The mapping $w=z^2$ takes each line $\Im(z)=c$ into a parabola; when $c=0$ this is the degenerate parabola represented by the positive u-axis covered twice. As $0\leq c\leq 1$ these parabolas fill the shaded region in the picture here, the "inside" of the parabola $u=(\frac{v}{2})^2-1$. As $1\leq c\leq\infty$ the parabolas fill in the outside of the shaded area. So the closed upper half-plane $\Im(z)\geq 0$ maps onto the entire (u,v)-plane, with the positive u-axis covered twice.

Since $(-z)^2=z^2$ the same thing happens for negative imaginary values: The region $-1\leq y\leq 0$ maps to the shaded area, and the region $-\infty\leq y\leq -1$ fills in the outside of the shaded area.

Depending on the form of the test you had, the answers were:

- $\Im(y) \geq 1$ maps to the outside of the shaded region.
- $\Im(y) \geq -1$ covers the whole plane. (Shaded region gets covered twice).
- $\Im(y) \leq 1$ covers the whole plane. (Shaded region gets covered twice).
- $\Im(y) \leq -1$ maps to the outside of the shaded region.

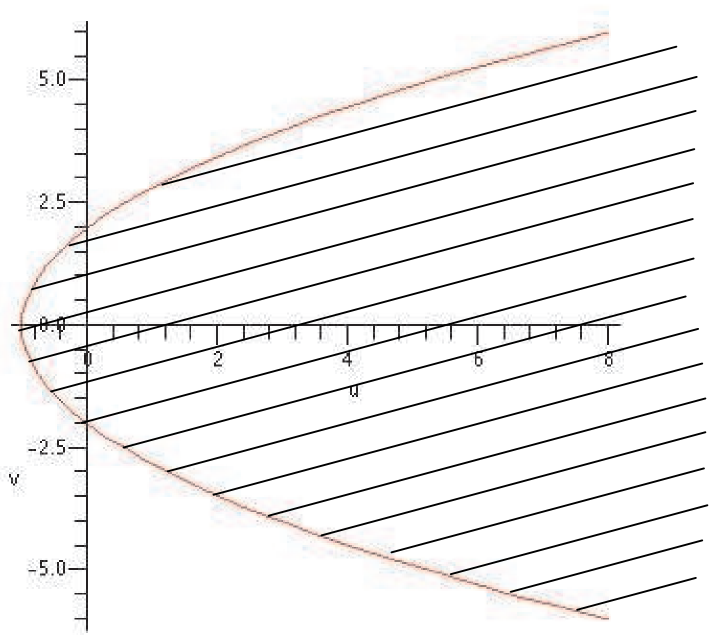

Figure 1: The parabola $u = (v/2)^2 - 1$ is the image of the line $\Im(z) = 1$ [and also of the line $\Im(z) = -1$].

3. (a) (15 points) Show carefully by an ϵ, δ argument that

$$\lim_{z \to a} \frac{f(z)}{g(z)} = 0$$

if $\lim_{z \to a} f(z) = 0$ and if there exists a pair of positive numbers δ_0, M such that $|z - a| < \delta_0$ implies $|g(z)| \geq M$.

Part (a) was almost identical to Problem 9 page 54, which we went over twice in class:

Given ϵ:

Since $\lim_{z\to a} f(z) = 0$ there is a δ_1 such that $|z-a| < \delta_1$ guarantees $|f(z)| < \epsilon M$.

Now take $\delta = \min(\delta_0, \delta_1)$. If $|z-a| < \delta$, then $|f(z)| < \epsilon M$ and $|g(z)| \geq M$; hence

$$|\frac{f(z)}{g(z)}| = \frac{|f(z)|}{|g(z)|} < \frac{\epsilon M}{M} = \epsilon.$$

(b) (10 points)Apply this to prove that

$$\lim_{z\to 0} \frac{z}{2+\frac{\overline{z}}{z}} = 0.$$

Since $\lim z \to 0z = 0$, it is enough by part (a) to find a δ_0 and an M that work for the denominator $2+\frac{\overline{z}}{z}$. (You need something like this, because $\lim_{z\to 0} \frac{\overline{z}}{z}$ *does not exist*). The triangle inequality ("backwards") gives

$$|2+\frac{\overline{z}}{z}| \geq 2 - |\frac{\overline{z}}{z}| = 2 - \frac{|\overline{z}|}{|z|} = 2-1 = 1.$$

So any positive number works for δ_0 with $M = 1$.