Stony Brook University MAT 341 Fall 2011
Homework Solutions, Chapter 4

§4.2 # 6 Solve the potential equation in the rectangle $0 < x < a, \quad 0 < y < b$, subject to the boundary conditions $u(a, y) = 1, 0 < y < b$, and $u = 0$ on the rest of the boundary.

SOLUTION: The boundary conditions are homogeneous with respect to y so when we separate variables $u(x, y) = X(x)Y(y)$,

$$\nabla^2 u = X''Y + XY'' = 0, \quad \frac{X''}{X} + \frac{Y''}{Y} = 0, \quad \frac{Y''}{Y} = -\frac{X''}{X} = c$$

(with boundary conditions $Y(0) = Y(b) = 0$ and $X(0) = 0$) the constant c should be negative, i.e. $-\lambda^2$. In that case the general solution for Y is

$$Y(y) = a_\lambda \cos(\lambda x) + b_\lambda \sin(\lambda x).$$

The boundary conditions on Y force first $a_\lambda = 0$ and then $\lambda = \frac{n\pi}{b}$: the n-th eigenfunction is $\sin \frac{n\pi y}{b}$.

The n-th corresponding X-equation is $X'' = (\frac{n\pi}{b})^2 X$ with general solution

$$X_n(x) = A_n \cosh \frac{n\pi}{b} x + B_n \sinh \frac{n\pi}{b} x.$$

The boundary condition $X(0) = 0$ forces $A_n = 0$.

Since these conditions are homogeneous, and the equation is linear, we can use superposition to write

$$u(x, y) = \sum_1^\infty B_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

and use the boundary condition $u(a, y) = 1$ to determine the coefficients $\{B_n\}$. In fact

$$u(a, y) = \sum_1^\infty B_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} = 1$$

means that $B_n \sinh \frac{n\pi a}{b}$ is the n-th Fourier sine coefficient of the function equal to 1 on $[0, b]$, i.e. the n-th Fourier coefficient of the square wave: $\frac{4}{\pi n}$ if n is odd, 0 if n is even. So if n is odd, then

$$B_n = \frac{4}{\pi n} \frac{1}{\sinh \frac{n\pi a}{b}}$$

and $B_n = 0$ if n is even.

§4.3 # 1 Solve the problem consisting of the potential equation on the rectangle $0 < x < a,\quad 0 < y < b$ with the given boundary conditions. Two of the three are very easy if a polynomial is subtracted from u.

(a). $\frac{\partial u}{\partial x}(0, y) = 0$; $u = 1$ on the rest of the boundary.

SOLUTION. Follow hint, set $v(x, y) = u(x, y) - 1$. Then $\nabla^2 v = \nabla^2 u = 0$ and the boundary conditions become $\frac{\partial v}{\partial x}(0, y) = 0$; $u = 0$ on the rest of the boundary. The solution to this problem is clearly the constant function $v = 0$, so the solution to the given problem is the constant function $u(x, y) = 1$.

(b). $\frac{\partial u}{\partial x}(0, y) = 0$; $\frac{\partial u}{\partial x}(a, y) = 0$; $u(x, 0) = 0$; $u(x, b) = 1$.

SOLUTION. Follow hint, set $v(x, y) = u(x, y) - \frac{y}{b}$. Then Then $\nabla^2 v = \nabla^2 u = 0$ and the boundary conditions become $\frac{\partial v}{\partial x}(0, y) = 0$; $\frac{\partial u}{\partial x}(a, y) = 0$; $u(x, 0) = 0$; $u(x, b) = 0$. The solution to this problem is clearly the constant function $v = 0$, so the solution to the given problem is the constant function $u(x, y) = \frac{y}{b}$.

(c). $\frac{\partial u}{\partial x}(x, 0) = 0$; $u(x, b) = 0$; $u(0, y) = 1$; $u(a, y) = 0$.

SOLUTION. The condition $\frac{\partial u}{\partial x}(x, 0) = 0$ is equivalent to $u(x, 0) = C$, a constant. Now the problem splits into two problems: $u = u_1 + u_2$ where

$$\nabla^2 u_1 = 0, \quad u_1(0, y) = u_1(a, y) = 0, \quad u_1(x, 0) = C, u_1(x, b) = 0$$

$$\nabla^2 u_2 = 0, \quad u_2(x, 0) = u_1(x, b) = 0, \quad u_2(0, y) = 1, u_2(a, y) = 0.$$

These each can be solved by the method of §4.2 # 6. To simplify the calculations switch to $v_1(x, y) = u_1(x, b - y)$ and $v_2(x, y) = u_2(a - x, y)$ and then switch back.

§4.4 # 4 Solve the potential problem in the slot $0 < x < a, 0 < y$, for each of these sets of boundary conditions.

(a.) $u(0, y) = 0$, $u(a, y) = 0$, $0 < y$; $u(x, 0) = 1, 0 < x < a$.

SOLUTION: This is a u_1-type problem (homogeneous x-boundary conditions) as on p. 279. So when we separate to get

$$\frac{X''}{X} = -\frac{Y''}{Y} = c$$

we should take $c = \lambda^2$; the X-eigenfunctions are then $\sin \frac{n\pi x}{a}$ as usual. The Y_n equation then has general solution $A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}$. The requirement that solutions be bounded as $y \to \infty$ forces $A_n = 0$. *Note how the choice of exponential solutions rather than hyperbolic-trigonometric simplifies the calculation. The other choice is also legitimate, but will involve more work.* The general solution is then

$$u(x,y) = \sum_0^\infty B_n \sin \frac{n\pi x}{a} e^{-\frac{n\pi y}{a}}.$$

The coefficients B_n are determined by the boundary condition $u(x,0) = 1, 0 < x < a$:

$$\sum_0^\infty B_n \sin \frac{n\pi x}{a} = 1.$$

This is the sine series for the square wave: $B_n = \frac{4}{\pi n}$ if n is odd, 0 if n is even.

(b.) $u(0,y) = 0, \quad u(a,y) = e^{-y}, \quad 0 < y; \; u(x,0) = 0, \quad 0 < x < a.$

SOLUTION. Here the boundary condition in y is homogeneous, so we separate as

$$\frac{X''}{X} = -\frac{Y''}{Y} = c$$

with c positive, e.g. $c = \lambda^2$, and we solve for Y first. The general Y solution is $Y = a\cos\lambda y + b\sin\lambda y$. The boundary condition $u(x,0) = 0$ forces $a = 0$. The corresponding X-equation is $X'' = \lambda^2 X$ with general solution $X = A(\lambda)\cosh\lambda x + B(\lambda)\sinh\lambda x$, and the general $u(x,y)$ solution is

$$u(x,y) = \int_0^\infty (A(\lambda)\cosh\lambda x + B(\lambda)\sinh\lambda x)\sin\lambda y \; d\lambda.$$

The coefficient functions $A(\lambda)$ and $B(\lambda)$ are determined by the boundary conditions $u(0,y) = 0$, $u(a,y) = e^{-y}$:

$$u(0,y) = \int_0^\infty A(\lambda)\sin\lambda y \; d\lambda = 0$$

means that $A(\lambda)$ corresponds to the Fourier sine integral of the zero function, so by uniqueness $A(\lambda) \equiv 0$. Then

$$u(a,y) = \int_0^\infty B(\lambda)\sinh\lambda a \sin\lambda y \; d\lambda = e^{-y}$$

means that $B(\lambda)\sinh\lambda a$ is the Fourier sine integral of e^{-y} so

$$B(\lambda)\sinh\lambda a = \frac{2}{\pi}\int_0^\infty e^{-y}\sin\lambda y \; dy = \frac{2}{\pi}\frac{\lambda}{1+\lambda^2}$$

(*integrate by parts or use table of integrals*) and

$$B(\lambda) = \frac{2}{\pi}\frac{\lambda}{1+\lambda^2}\frac{1}{\sinh\lambda a}.$$

(c.) Similar to (b.), with an easier integral. Make life simpler by setting $v(x,y) = u(a-x,y)$ so $v(0,y) = 0$ and $v(a,y) = f(y)$; switch back to u to finish.

§4.4 # 5 Solve the potential problem in the slot $0 < x < a, 0 < y$, for each of these sets of boundary conditions.

(a.) $\frac{\partial u}{\partial x}(0,y) = 0, u(a,y) = 0, 0 < y$; $u(x,0) = 1, 0 < x < a$.

SOLUTION. In this case the x-problem is homogeneous, so we separate as

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

and solve for X first. The general solution is $X(x) = a\cos\lambda x + b\sin\lambda x$; the boundary conditions translate to $X'(0) = 0$, $X(a) = 0$. The first forces $b = 0$; the second forces $\lambda = \frac{(2n-1)\pi}{2a}$, so the n-th X-eigenfunction is $\cos\frac{(2n-1)\pi x}{2a}$. The corresponding Y_n equation is

$$Y_n'' = (\frac{(2n-1)\pi}{2a})^2 Y_n$$

with solution $Y_n = a_n\exp(\frac{(2n-1)\pi y}{2a}) + b_n\exp(-\frac{(2n-1)\pi y}{2a})$. *Note choice of basis for solutions.* The requirement that solutions be bounded as $y \to \infty$ forces $a_n = 0$. The general solution is then

$$u(x,y) = \sum_0^\infty b_n e^{-\dfrac{(2n-1)\pi y}{2a}} \cos\frac{(2n-1)\pi x}{2a}$$

where the $\{b_n\}$ are determined by the boundary condition

$$u(x,0) = \sum_0^\infty b_n \cos \frac{(2n-1)\pi x}{2a} = 1.$$

The functions $\cos \frac{(2n-1)\pi x}{2a}$ for $n = 1, 2, 3, ...$ are an orthogonal family on $[0, a]$ with

$$\int_0^a \cos^2 \frac{(2n-1)\pi x}{2a}\, dx = \frac{a}{2},$$

so

$$b_n = \frac{2}{a}\int_0^a \cos \frac{(2n-1)\pi x}{2a}\, dx = \frac{2}{a}\frac{2a}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{2a}|_0^a$$

$$= \frac{2}{a}\frac{2a}{(2n-1)\pi} \sin \frac{(2n-1)\pi}{2} = \begin{cases} \frac{4}{(2n-1)\pi} & n = 1, 3, 5, ... \\ -\frac{4}{(2n-1)\pi} & n = 2, 4, 6, ... \end{cases}.$$

(b.) $\frac{\partial u}{\partial x}(0, y) = 0, u(a, y) = e^{-y}, 0 < y$; $u(x, 0) = 0, 0 < x < a$. Here the y-boundary condition is homogeneous, so we separate as

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2$$

and solve for Y first. The general solution is $Y(y) = a \cos \lambda y + b \sin \lambda y$; the boundary condition translates to $Y(0) = 0$, which forces $a = 0$. The X equation is then $\frac{X''}{X} = \lambda^2$, with general solution $X = A(\lambda) \cosh \lambda x + B(\lambda) \sinh \lambda x$. The general solution for u is

$$u(x, y) = \int_0^\infty (A(\lambda) \cosh \lambda x + B(\lambda) \sinh \lambda x) \sin \lambda y \; d\lambda,$$

where the coefficient functions are determined by the boundary conditions using Fourier integrals. Namely:

$$\frac{\partial u}{\partial x}(0, y) = \int_0^\infty (\lambda B(\lambda) \sin \lambda y \; d\lambda = 0$$

means that $\lambda B(\lambda)$ is the Fourier sine integral for the zero function; by uniqueness $\lambda B(\lambda) \equiv 0$ so $B(\lambda) \equiv 0$. Then

$$u(a,y) = \int_0^\infty A(\lambda)\cosh\lambda a \sin\lambda y \; d\lambda = e^{-y}$$

means that $A(\lambda)\cosh\lambda a$ is the Fourier sine integral for e^{-y}, i.e.

$$A(\lambda)\cosh\lambda a = \frac{2}{\pi}\int_0^\infty e^{-y}\sin\lambda y \; dy = \frac{2}{\pi}\frac{\lambda}{1+\lambda^2}$$

(the same integral as in 4.b.) so

$$A(\lambda) = \frac{2}{\pi}\frac{\lambda}{1+\lambda^2}\frac{1}{\cosh\lambda a}.$$

(c.) $u(0,y) = 0, u(a,y) = f(y) = \begin{cases} 1, & 0 < y < b \\ 0 & b < y \end{cases}$, $\frac{\partial u}{\partial y}(x,0) = 0, 0 < x < a$.

SOLUTION: Here the y-conditions are homogeneous, so we separate and set $\frac{Y''}{Y} = -\lambda^2$, $\frac{X''}{X} = \lambda^2$, and solve for Y first. The general solution is $Y = a\cos\lambda y + b\sin\lambda y$; the boundary condition at $y = 0$ translates to $Y'(0) = 0$, which forces $b = 0$, with no condition on λ. The corresponding X equation has general solution $X(x) = A(\lambda)\cosh\lambda x + B(\lambda)\sinh\lambda x$, leading to

$$u(x,y) = \int_0^\infty (A(\lambda)\cosh\lambda x + B(\lambda)\sinh\lambda x)\cos\lambda y \; d\lambda$$

where $A(\lambda)$ and $B(\lambda)$ are determined by the boundary conditions:

$$u(0,y) = \int_0^\infty A(\lambda)\cos\lambda y \; d\lambda = 0$$

gives $A(\lambda)$ as the cosine integral of the zero function, so $A(\lambda) \equiv 0$. Then

$$u(a,y) = \int_0^\infty B(\lambda)\sinh\lambda a \cos\lambda y \; d\lambda = f(y)$$

gives

$$B(\lambda)\sinh\lambda a = \frac{2}{\pi}\int_0^\infty f(y)\cos\lambda y \; dy = \frac{2}{\pi}\int_0^b \cos\lambda y \; dy = \frac{2}{\pi\lambda}\sin\lambda b$$

and so

$$B(\lambda) = \frac{2}{\pi\lambda}\frac{\sin\lambda b}{\sinh\lambda a}.$$

§4.5 # 1 Solve the potential equation in the disc $0 < r < c$ if the boundary condition is $v(c, \theta) = |\theta|$, $-\pi < \theta < pi$.

SOLUTION. As described in §4.5, the potential equation $\nabla^2 v = 0$ leads, via writing $v(r, \theta) = R(r)Q(\theta)$, to $Q'' = -\lambda^2 Q$ with general solution $Q(\theta) = a\cos\lambda\theta + b\sin\lambda\theta$; since Q must be periodic of period 2π for the function to be well-defined on the disc, λ must be an integer: $n = 0, 1, 2, 3, ...$ (negative integers don't give new solutions). The corresponding R_n must satisfy the Cauchy-Euler equation; the solutions are $R_n(r) = r^n$, $R_n(r) = r^{-n}$. The second solution blows up at $r = 0$ and is not useful. The solution to the problem is then

$$v(r, \theta) = a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) r^n,$$

where the coefficients a_n, b_n are determined from the initial conditions by Fourier analysis:

$$v(c, \theta) = a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) c^n = |\theta|.$$

So a_0, $a_n c^n$ and $b_n c^n$ are the coefficients of the Fourier series of $f(\theta) = |\theta|, \quad -\pi < \theta < \pi$. This f is an even function, so the sine coefficients are zero, and

$$a_0 = \frac{1}{\pi}\int_0^\pi \theta \, d\theta = \frac{\pi}{2}$$

$$a_n c^n = \frac{2}{\pi}\int_0^\pi \theta \cos n\theta \, d\theta = \begin{cases} \dfrac{-4}{\pi n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(note that $|\theta| = \theta$ on $[0, \pi]$). Finally

$$v(r, \theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{r^n \cos n\theta}{n^2 c^n}.$$

§4.5 #4 Same as Exercise 1 with boundary condition

$$v(c, \theta) = f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}.$$

SOLUTION. Same as Exercise 1, except here f is odd, so the cosine coefficients are zero, and

$$b_n c^n = \frac{2}{\pi}\int_0^\pi \sin n\theta\ d\theta = -\frac{2}{\pi n}\cos n\theta|_0^\pi = \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

So

$$v(r,\theta) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{nc^n} r^n \sin n\theta.$$

§4.5 #5 Find the value of the solution at $r = 0$ for the problems of Exercises 1 and 4.

SOLUTION. When $r = 0$ the solution of Exercise 1 gives $v = \frac{\pi}{2}$, and the solution of Exercise 4 gives $v = 0$.