Stony Brook University MAT 341 Fall 2011
Homework Solutions, Chapter 3.
(Some notation changed on Nov 11 at 5:45 PM).

§3.2 # 5 Solve the vibrating-string problem with the initial conditions $f(x) = 0, \quad g(x) = 1, \quad 0 < x < a$.

SOLUTION: The general solution for a vibrating string of length a is (Equation (9))

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{a}x)[a_n \cos(\frac{n\pi}{a}ct) + b_n \sin((\frac{n\pi}{a}ct)].$$

The initial conditions $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$ determine a_n and b_n by Fourier analysis (pp. 220-221). Here $f(x) = 0$ so

$$a_n = \frac{2}{a}\int_0^a f(x)\sin(\frac{n\pi}{a}x)\ dx = 0.$$

Also in this problem $g(x) = 1$ so

$$\begin{aligned} b_n &= \frac{2}{n\pi c}\int_0^a g(x)\sin(\frac{n\pi}{a}x)\ dx = \frac{2}{n\pi c}\int_0^a \sin(\frac{n\pi}{a}x)\ dx \\ &= \frac{2}{n\pi c}\frac{a}{n\pi}(-\cos(\frac{n\pi}{a}x))|_0^a = \frac{2a}{n^2\pi^2 c}(1-\cos n\pi) \\ &= \begin{cases} \frac{4a}{n^2\pi^2 c} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}. \end{aligned}$$

§3.3 # 7 Solve (pressure of air in organ pipe)

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \qquad 0 < x < a, \quad 0 < t$$

with boundary conditions:

a. $p(0,t) = p_0$, $p(a,t) = p_0$.

SOLUTION: to have homogeneous boundary conditions we define $q(x,t) = p(x,t) - p_0$. Then q satisfies the same wave equation as p, but $q(0,t) = 0$,

$q(a,t)=0$. Separating variables as usual, we set $q(x,t)=\phi(x)T(t)$ and derive

$$\frac{\phi''}{\phi}=\frac{1}{c^2}\frac{T''}{T}=K.$$

The boundary conditions on q give boundary conditions $\phi(0)=0, \phi(a)=0$ which can only be satisfied by a non-zero ϕ if K is negative, say $K=-\lambda^2$. Then the general solution is $\phi(x)=A\cos(\lambda x)+B\sin(\lambda x)$. The boundary condition $\phi(0)=0$ gives us $A=0$. Then the boundary condition $\phi(a)=0$ gives $B\sin\lambda a=0$, so $\lambda=\frac{n\pi}{a}$. These are the eigenvalues, and $\phi_n=\sin\frac{n\pi}{a}x$ are the eigenfunctions.

a. $p(0,t)=p_0$, $\frac{\partial p}{\partial x}(a,t)=0$.

Again we set $q(x,t)=p(x,t)-p_0$, and separate variables as before. The boundary conditions on ϕ are now $\phi(0)=0$ and $\phi'(a)=0$. (As before, these force $K=-\lambda^2$.) The general solution is again $\phi(x)=A\cos(\lambda x)+B\sin(\lambda x)$. Again $\phi(0)=0$ forces $A=0$. Now $\phi=B\sin\lambda x$ and $\phi'(x)=\lambda B\cos\lambda x$. Setting $\phi'(a)=0$ gives $\cos\lambda a=0$ so λ must be an odd multiple of $\frac{\pi}{2a}$. These are the eigenvalues: $\frac{\pi}{2a}$, $\frac{3\pi}{2a}$, $\frac{5\pi}{2a}$, etc. The corresponding eigenfunctions are $\sin\frac{\pi}{2a}x$, $\sin\frac{3\pi}{2a}x$, $\sin\frac{5\pi}{2a}x$, etc.

§3.2 # 9 Find eigenfunctions, eigenvalues and product solutions for:

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}+k\frac{\partial u}{\partial t}, \qquad 0<x<a, \quad 0<t$$

$$u(0,t)=0, \quad u(a,t)=0, \qquad t<0$$

SOLUTION: Separation of variables leads to

$$\frac{\phi''}{\phi}=\frac{1}{c^2}\frac{T''}{T}+k\frac{T'}{T}=K$$

where the boundary conditions on ϕ force $K=-\lambda^2$ as usual, and as usual we find eigenvalues $\lambda_n=\frac{n\pi}{a}$ and eigenfunctions $\phi_n(x)=\sin\frac{n\pi}{a}x$, $n=1,2,3,....$ The corresponding T problem is

$$\frac{1}{c^2}\frac{T_n''}{T_n}+k\frac{T_n'}{T_n}=-\lambda_n^2,$$

or

$$T_n'' + c^2 k T_n' + c^2 \lambda_n^2 T_n = 0.$$

This is a linear order-2 equation with constant coefficients. As we learned in Calc II, we look for a solution of the form $T_n = e^{a_n t}$. Substituting this form in the differential equation yields

$$a_n^2 + c^2 k a_n + c^2 \lambda_n^2 = 0,$$

an algebraic equation with solutions:

$$a_n = \frac{-c^2 k \pm \sqrt{c^4 k^2 - 4c^2 \lambda_n^2}}{2}.$$

We are told that k is small, so we can take the square root to be imaginary, and the two solutions we get are

$$T_n = e^{-\frac{c^2 kt}{2}} e^{\frac{i\sqrt{4c^2\lambda_n^2 - c^4 k^2}}{2} t}, T_n = e^{-\frac{c^2 kt}{2}} e^{\frac{-i\sqrt{4c^2\lambda_n^2 - c^4 k^2}}{2} t}.$$

As usual, any combination of $e^{i\omega t}$ and $e^{-i\omega t}$ can be rewritten as a combination of $\sin \omega t$ and $\cos \omega t$; so the most general product solution is

$$u_n(x,t) = \sin \lambda_n x \, e^{-\frac{c^2 kt}{2}} [a_n \cos(\frac{c}{2}\sqrt{4\lambda_n^2 - c^2 k^2}\, t) + b_n \sin(\frac{c}{2}\sqrt{4\lambda_n^2 - c^2 k^2}\, t)],$$

where $\lambda_n = \frac{n\pi}{a}$.