Stony Brook University - MAT 341 Final Examination

December 22, 2005

This test is open book: Powers "Boundary Value Problems" may be consulted. No other references or notes may be used.

Students may use graphing calculators like TI-83, 85, 86; but they may NOT use calculators with Computer Algebra Systems, like TI-89.

Total score = 140.

1. (a) (20 points) Calculate the Fourier series for the function $f(x)$, periodic of period 2π, equal to to 0 on $[0, \pi/2]$, to 1 on $[\pi/2, 3\pi/2]$ and to 0 on $[3\pi/2, 2\pi]$.

   (b) (8 points) Use this Fourier series to deduce the following series representation for π:

   $$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots).$$

2. (a) (20 points) A cooper bar ($k = 1.1$ cm^2/sec) of length 20 cm, insulated along its length, is initially at temperature 100° C. Starting at time $t = 0$ both ends of the bar are kept at 0° C. Solve the heat equation to calculate the temperature $u(x, t)$ at a point in the bar x cm from the left end and at a time $t > 0$.

   (b) (8 points) At what time will the temperature at the midpoint of the bar be equal to 50° C? Note that for $t > 0$ the first term in the series expansion for the solution is much larger than the sum of the others.

3. The motion of a string of length a, fixed at both ends and vibrating *in a viscous medium* is governed by the equation

   $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}(\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t})$$

   (the damped wave equation) with boundary conditions

   $$u(0, t) = 0, u(a, t) = 0$$

where k measures the viscosity of the medium ($k = 0$ is the standard wave equation) and c is the velocity of propagation.

(a) (20 points) Set $u(x,t) = \phi(x)T(t)$ as usual to separate variables; set up and solve the associated eigenvalue problem for ϕ.

(b) (4 points) Give the general solution to the damped wave equation, assuming that k is smaller than $\pi c/a$.

(c) (4 points) What happens when $k = \pi c/a$?

4. A disk of radius 1 is completely insulated (both faces and around the boundary). The initial temperature distribution is

$$u(r,\theta,0) = \begin{cases} 1 & 0 \le r \le \frac{1}{2} \\ 0 & \frac{1}{2} < r \le 1 \end{cases}, \quad 0 \le \theta < 2\pi.$$

(a) (20 points) What is the steady-state temperature distribution?

(b) (8 points) Note that by symmetry the θ-derivatives of the solution function must be zero. Separate variables as $u(r,\theta,t) = R(r)T(t)$, set up and solve the eigenvalue problem for R.

5. (a) (20 points) Show that the functions $u(r,\theta) = r^n \cos(n\theta)$ and $u(r,\theta) = r^n \sin(n\theta)$ satisfy the potential equation in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

(b) (8 points) Use this information to solve the Plateau problem in a disk of radius 1

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 \le r < 1, \quad 0 \le \theta < 2\pi$$

with boundary condition

$$u(1,\theta) = \begin{cases} 1 & 0 \le \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}, \quad 0 \le \theta < 2\pi.$$