Stony Brook University - MAT 320 Midterm II

Solutions

1. (25 points) The functions f and g are continuous on the interval $[a, b]$ and differentiable on (a, b). If $f(a) = g(a)$ and $f(b) = g(b)$, prove that there is a point c, with $a < c < b$, where $f'(c) = g'(c)$.

   - Apply Rolle's Theorem to $f - g$.

2. (25 points) The function $\sin x$ is infinitely differentiable on $\mathbf{R}$, and its derivatives cycle through $\cos x, -\sin x, -\cos x, \sin x$. Let $P_k(x)$ be the k-th Taylor polynomial, based at 0, for $\sin x$. Prove that

$$\lim_{k\to\infty} P_k(x) = \sin x$$

   for every $x \in \mathbf{R}$.

   - Choose x. By Taylor's Theorem, there exists c between 0 and x such that
$$\sin x - P_k(x) = \frac{f^{(k+1)}(c)x^{k+1}}{(k+1)!}.$$
     Since $|f^{(k+1)}(c)| \le 1$, we have $|\sin x - P_k(x)| \le |x|^{k+1}/(k+1)!$. The proof then follows from $\lim_{n\to\infty} a^n/n! = 0$, true for any $a \in \mathbf{R}$.

3. (25 points) The function f is continuous and twice differentiable on the interval $[a, b]$, with both derivatives positive there: $f'(x) > 0$ and $f''(x) > 0$ for every $a \le x \le b$. Suppose that $f(a) < 0$ and $f(b) > 0$. Prove that the tangent line to the graph of f at b intersects the x-axis at a point in the interval $[a, b]$. (This corresponds to starting Newton's Method with $x_0 = b$, and then x_1 is that intersection point, but no knowledge of Newton's Method is necessary for this question).

   - Since Let x be the intersection point; since $f(b) > 0$ and $f'(b) > 0$, we know $x < b$. To show $a < x$, argue as follows: by the Intermediate Value Theorem, there is c, $a < c < b$, with $f(c) = 0$. By the Mean Value Theorem, there is k, $c < k < b$, with $f'(k) = (f(b) - f(c))/(b - c)$, or $f'(k) = f(b)/(b - c)$. The equation of the tangent line at $(b, f(b)$ is $y = f(b) + f'(b)(x - b)$, and it intersects the x-axis when $x - b = -f(b)/f'(b)$. By hypothesis $f''(x)$ is always positive, so $f'(k) < f'(b)$. Finally

$$b - x = f(b)/f'(b) < f(b)/f'(k) = b - c.$$

It follows that $c < x$, and therefore $a < x$.

4. (25 points) The Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ... are defined by the properties $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Define a sequence (x_n) by

$$x_n = \frac{F_{n+1}}{F_n}.$$

Prove that this sequence converges, and calculate its limit.

- Notice first that $x_{n+1} = 1 + 1/x_n$. We first use this to show $x_n \geq 3/2$ as soon as $n \geq 1$: the x_n are all < 2 (since $F_{n+1} = F_n + F_{n-1} < 2F_n$); so $x_{n+1} = 1 + 1/x_n > 1 + 1/2$. Next we use it to show the sequence (x_n) is contractive:

$$|x_{n+1} - x_n| = |1 + 1/x_n - 1 - 1/x_{n-1}| = \frac{|x_{n-1} - x_n|}{x_n x_{n-1}} \leq (4/9)|x_{n-1} - x_n|$$

using the first calculation. It follows that the sequence converges to a limit L. Substituting L in $x_{n+1} = 1 + 1/x_n$ and remembering that the x_n are all positive leads to $L = (1/2)(1 + \sqrt{5})$.