Correction of HW5

Exercise 1. Page 80, #1.

Proof.
Take for example the following sequence: $x_{2k}=k, x_{2k+1}=1$.

$\square$

Exercise 2. Page 80, #3.

Proof. Recall that f_n satisfies the relation $f_{n+2}=f_{n+1}+f_n$ and that all the f_n are >0.
therefore we deduce that $\frac{f_{n+2}}{f_{n+1}}=1+\frac{f_n}{f_{n+1}}$.We know that $x_n=\frac{f_{n+1}}{f_n}$ has a limit L. Notice that L cannot be zero (because then $\frac{f_n}{f_{n+1}}$ would then be unbounded, which is not the case because it is equal to $\frac{f_{n+2}}{f_{n+1}}-1$, which converges to $L-1$). Thus we can apply the quotient theorem and obtain the equality $L=1+\frac{1}{L}$. This implies that $L^2=L+1$.By solving this equation and keeping the positive root we get $L=\frac{1+\sqrt{5}}{2}$.

$\square$

Exercise 3. Page 80, #8a.

Proof. Let's compare $x_{n+1}=(3(n+1))^{1/2(n+1)}$ and $x_n=(3n)^{1/2n}$. We have
$x_{n+1}^{2n.2(n+1)}=3(n+1)^{2n}=(3n)^{2n}.(1+\frac{1}{n})^{2n}$, whereas $x_n^{2n.2(n+1)}=(3n)^{2(n+1)}=(3n)^{2n}.(3n)^2$, therefore after some integer K, the sequence is decreasing (because $(1+\frac{1}{n})^{2n}$ is eventually strictly smaller than $(3n)^2$). Since it is bounded below by 0, it has a limit L.
Now the subsequence x_{2n} must converge to the same limit L, but we have
$x_{2n}=(3.2n)^{1/4n}=2^{1/4n}.x_n^{1/2}$ so this converges to $1.\sqrt{L}=\sqrt{L}$ so $L=\sqrt{L}$ and thus $L=1$.

$\square$

Exercise 4. Page 80, #14.

Proof. Pick $\varepsilon_1=1$, then we know the existence of x_{n_1} such that $s-\varepsilon_1<x_{n_1}\leqslant s$ (def. of a sup), and we even know that $x_{n_1}<s$.
Pick $\varepsilon_2=\frac{1}{k_2}$, such that k_2 is at least 2 (thus $\varepsilon_2<\frac{1}{2}$), and such that $x_{n_1}<s-\varepsilon_2$, then one knows the exisence of x_{n_2} such that $s-\varepsilon_2<x_{n_2}<s$.
Pick $\varepsilon_3=\frac{1}{k_3}$, such that k_3 is at least 3 (thus $\varepsilon_3<\frac{1}{3}$), and such that $x_{n_2}<s-\varepsilon_3$, then one knows the existence of x_{n_3} such that $s-\varepsilon_3<x_{n_3}<s$.
By continuing like this one constructs an increasing subsequence x_{n_k} that has the property that $s-\frac{1}{k}<x_{n_k}<s$, therefore it converges to s (Squeeze theorem!). $\square$

Exercise 5. Page 80, #15.

Proof. Since the I_n are nested one knows that $x_n \in I_0$ so this sequence is bounded and therefore by Bolzano-Weierstrass. it has a converging subsequence (x_{n_k}), with limit L.

Let's prove by contradiction that $L \in \cap_{n=1}^{\infty} I_n$. Indeed, if it's not the case, then say $L \notin I_N$ for some N. Pick $\varepsilon > 0$ small enough such that $(L-\varepsilon, L+\varepsilon) \cap I_N = \emptyset$. By convergence of (x_{n_k}), there is some element of this subsequence, say x_{n_K} that lands in $(L-\varepsilon, L+\varepsilon)$ and such that n_K is larger than N, thus I_{n_K} intersects $(L-\varepsilon, L+\varepsilon)$, but this is a contradiction because $I_{n_K} \subset I_N$ (and I_N doesn't intersect that interval).

□

Exercise 6. Page 86, #1.

Proof. $(-1)^n$ is bounded and not convergent so it's not a Cauchy sequence.

□

Exercise 7. Page 86, #3c.

Proof. $(\ln n)$ is not bounded so it's certainly not a Cauchy sequence. This can be proved using the definition:pick $\varepsilon = 1$, can we find K such that for any $n, m \geqslant K$ one has $|\ln n - \ln m| = |\ln \frac{n}{m}|$ less than 1? The answer is no: take $n = 5m > m \geqslant K$, then $\ln \frac{6m}{m} = \ln 6 > 1$. □

Exercise 8. Page 86, #9.

Proof. Notice that $|x_{n+p} - x_n| \leqslant |x_{n+p} - x_{n+p-1}| + \ldots + |x_{n+1} - x_n| < r^{n+p-1} + \ldots + r^n$.

But this last sum is also $r^n.(r^{p-1} + \ldots + 1) = r^n.\frac{1-r^p}{1-r} < \frac{1}{1-r}.r^n$.

Given any $\varepsilon > 0$, then one can find a natural number K such that for any $n \geqslant K$ one has $\frac{1}{1-r}.r^n \leqslant \frac{1}{1-r}.r^K < \varepsilon$, and thus the sequence is a Cauchy sequence.

□

Exercise 9. Page 86, #13.

Proof. First we notice that x_n is never zero so the sequence is well-defined.

Then one has $\left|x_{n+2} - x_{n+1}\right| = \left|2 + \frac{1}{x_{n+1}} - 2 - \frac{1}{x_n}\right| = \left|\frac{x_n - x_{n+1}}{x_n.x_{n+1}}\right| \leqslant \frac{1}{4}.\left|x_{n+1} - x_n\right|$ because every x_n is > 2.So the sequence is contractive, and therefore converges to a limit x.This limit must satisfy $x = 2 + \frac{1}{x}$ and be positive, thus one must have $x^2 = 2x + 1$, and then one gets that

$x = 1 + \frac{\sqrt{5}}{2}$.

□