Correction of HW3

Exercise 1. Page 50, #2.

Proof.

1. If S is bounded then there exists a lower bound m and an upper bound M. By definition, they are such that any x in S satisfies $m \leqslant x \leqslant M$. But this means $x \in [m, M]$.Therefore $S \subset [m, M]$.
2. Conversely, $S \subset [m, M]$ exactly means that any x in S is bounded above by M, and below by m.

$\square$

Exercise 2. Page 50, #9.

Proof. By contradiction: assume that the intersection is non empty, and therefore contains some real number x. Pick any integer K strictly larger than x (for example $1 + E(x)$, where $E(x)$ is the integral part of x): then clearly $x \notin (K, \infty)$ and thus $x \notin \bigcap_{n=1}^{\infty} (n, \infty)$, a contradiction.

$\square$

Exercise 3. Page 50, #13.

Proof. Since 1/3 is strictly less than 1, the binary representation starts with 0.

We want to find a, b, c, $d \in \{0, 1\}$ such that the binary representation of 1/3 starts with $(0.\text{abcd}...)_2$.

We notice that $\frac{1}{2} > \frac{1}{3}$, so the first digit a must be 0 (not one). Then $\frac{1}{4} < \frac{1}{3}$, so the next digit is 1. Then $\frac{1}{4} + \frac{1}{8}$ is too large so the following digit must be 0. Similarly the fourth digit is 1 because $\frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} < 1/3$.

It seems that there is a pattern: so let's prove that the binary expansion of 1/3 is 0.010101...

Call $x := (0.0101010101...)_2$ Then notice that $2^2.x = (1.01010101...)_2$, so by subtraction one has that $(2^2 - 1).x = 1$ which means exactly that $x = 1/3$.

$\square$

Exercise 4. Page 50, #17.

Proof. Write $x = 1.25137137...$ then $100x = 125 + 0.137137...$

But if you write $y = 0.137137...$, you see that $999y = 137$, therefore $x = \frac{125 + \frac{137}{999}}{100} = \frac{125012}{99900}$.

Similarly, if $y = 35.14653653...$ you see that $100y = 3514 + \frac{653}{999}$ therefore $y = \frac{3511139}{99900}$.

$\square$

Exercise 5. Page 59, #3c.

Proof. We have already $z_1 = 1, z_2 = 2, z_3 = \frac{2+1}{2-1} = 3, z_4 = \frac{3+2}{3-2} = 5, z_5 = \frac{5+3}{5-2} = \frac{8}{3}$. □

Exercise 6. Page 59, #4.

Proof. Let $\varepsilon > 0$, by the archimedean property we know the existence of an integer K satisfying $K \geqslant \frac{|b|}{\varepsilon}$. Therefore, for any $n \geqslant K$ one has $n \geqslant \frac{|b|}{\varepsilon}$ and thus $\left|\frac{b}{n}\right| \leqslant \varepsilon$. Therefore, the sequence is converging to zero. □

Exercise 7. Page 59, #5c.

Proof. One has $0 \leqslant \left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{6n+2-6n-15}{4n+10}\right| \leqslant \frac{13}{4n+10} \leqslant \frac{13}{4}.\frac{1}{n}$

Since we know that $1/n$ converges to zero, we deduce that x_n converges to $3/2$. □

Exercise 8. Page 59, #6c.

Proof. One has $0 \leqslant \left|\frac{\sqrt{n}}{n+1}\right| \leqslant \frac{1}{\sqrt{n}}$ for $n \geqslant 1$, so it is enough to prove that $1/(\sqrt{n})$ converges to zero.

Given any $\varepsilon > 0$, by the archimedean property one can find an integer $K \geqslant \varepsilon^2$, therefore for any $n \geqslant K$, one has $n \geqslant \varepsilon^2$ and so $0 \leqslant 1/(\sqrt{n}) \leqslant \varepsilon$, so $(1/\sqrt{n})$ converges to zero. □

Exercise 9. Page 59, #8.

Proof. The convergence of (x_n) to zero translates as follows:

for any $\varepsilon > 0$ there exists an integer K such that: for all $n \geqslant K$ one has $|x_n| < \varepsilon$.

The convergence of $(|x_n|)$ to zero translates as follows:

for any $\varepsilon > 0$ there exists an integer K such that: for all $n \geqslant K$ one has $||x_n|| < \varepsilon$.

Since $|x_n| \geqslant 0$, one has that $|x_n| = ||x_n||$ so the two propositions are equivalent.

Now if $x_n = (-1)^n$, one can see that $|x_n| = 1$ so it converges, but (x_n) doesn't converge.

□

Exercise 10. Page 67, #5b.

Proof. A convergent sequence must be bounded. Since $((-1)^n.n^2)$ is unbounded, it cannot converge.

(Remark: to be convinced that it is unbounded, use the Archimedean property. Given any $M > 0$, there exists an integer $K > \sqrt{M}$ and therefore any $n \geqslant K$ satisfies $\left|(-1)^n.n^2\right| > M$)

□

Exercise 11. Page 67, #6d.

Proof. One has $x_n = \frac{n+1}{n\sqrt{n}} = \frac{1}{\sqrt{n}} + \frac{1}{n\sqrt{n}}$. Now we have already proved above that $(1/\sqrt{n})$ converges to zero (Archimedean property!), and since $0 \leqslant \left|\frac{1}{n\sqrt{n}}\right| \leqslant \frac{1}{n} \to 0$, we see that x_n is the sum of two sequences converging to zero, therefore it converges to zero.

□

Exercise 12. Page 67, #7.

Proof. Let $M > 0$ be an upper bound for the sequence (b_n).

Given any $\varepsilon > 0$, since (a_n) converges to zero, we know the existence of an integer K such that for all $n \geqslant K$ one has $\left|a_n\right| \leqslant \frac{\varepsilon}{M}$.

Now for any $n \geqslant K$, one has $|a_n.b_n| \leqslant |a_n|.M \leqslant \varepsilon$. But this exactly says that $(a_n b_n)$ converges to zero.

The theorem 3.2.3 cannot be applied because (b_n) is only bounded, and not necessarily convergent.

$\square$

Exercise 13. Page 67, #17.

Proof. Let r be a real number satisfying $1 < r < L$. Since $\left(x_{n+1}/x_n\right)$ converges to L, we know the existence of an integer K such that for any $n \geqslant K$ one has $\left|\frac{x_{n+1}}{x_n} - L\right| < L - r$. But this implies that for any $n \geqslant K$ one has $\frac{x_{n+1}}{x_n} > r$.

Let's prove by induction that for any $n \geqslant K$ one has $x_n \geqslant r^{n-K}.x_K$

This is true for $n = K$ because $x_K = r^0.x_K$.

Assume it is true for n, then we have that $x_{n+1} > r.x_n > r.r^{n-K}.x_K = r^{n+1-K}.x_K$, so we are done.

Now it remains to prove that the sequence (r^n) for $r > 1$ is unbounded.

Here is one possible way: write $r = 1 + d$, and prove by induction that for any n one has $(1+d)^n > 1 + n.d$.

Another way is to take the $\log(r^n)$ and apply the archimedean property.

$\square$