Correction of HW2

Exercise 1. Page 38, #2.

Proof. The set S_2 is not empty and is bounded below (for example by 0), so it has an infimum. Let's prove that $0 = \inf(S_2)$:

1. For any x in S_2, one has $0 \leqslant x$;
2. For any $\varepsilon > 0$, one can find x in S_2 such that $0 \leqslant x < 0 + \varepsilon$ (take $\varepsilon/2$ for example).

Therefore $0 = \inf(S_2)$.

Now S_2 is not bounded above, so it doesn't have upper bounds (and therefore doesn't have a sup).

□

Exercise 2. Page 38, #3.

Proof. Supremum: Since $n \in \mathbb{N} \Rightarrow \frac{1}{n} \leqslant 1$, one knows that S_3 is bounded above, since it is also a non empty subset of $\mathbb{R}$, we know that it has a supremum. Let's prove that $1 = \sup(S_3)$:

1. For any x in S_3, one has $x \leqslant 1$;
2. For any $\varepsilon > 0$, one can find an x in S_3 such that $1 - \varepsilon < x \leqslant 1$ (just take $x = 1$!)

Infimum:

Let's prove that $0 = \inf(S_3)$:

1. For any $x = 1/n$ in S_3, one has $x \geqslant 0$;
2. For any $\varepsilon > 0$, one can find an x in S_3 such that $0 \leqslant x < \varepsilon$ (indeed by the archimedean property one knows the existence of an integer $n_\varepsilon > 1/\varepsilon$, then just take $x = 1/n_\varepsilon$).

□

Exercise 3. Page 38, #7.

Proof. a) Assume that u is an upper bound of S non empty:
this means that for any x in S one has $x \leqslant u$. Now if t is any real number such that $t > u$, we will get that $t > x$ for any $x \in S$, so $t \notin S$.

b) Conversely: Assume now that $(t > u) \Rightarrow t \notin S$. Suppose that u is not an upper bound. Since S is not empty, this would imply the existence of $y \in S$ such that $y > u$ (contradiction).

□

Exercise 4. Page 38, #9.

Proof. a) If α is an upper bound for A, and β is an upper bound for B, then the maximum of the two numbers α, β is an upper bound for $A \cup B$. For the lower bounds, take the minimum instead.So the union of two bounded sets is a bounded set.

b) At this point we know the existence of sup $(A \cup B)$.Let's prove that sup $(A \cup B)$ = sup $\{\sup A, \sup B\}$:

1. We already know that $Z = \sup\{\sup A, \sup B\}$ is an upper bound of $A \cup B$;
2. For any $\varepsilon > 0$, is there an element $x \in A \cup B$ such that $Z - \varepsilon < x$?

There are two cases: if $Z = \sup A$, then we now the existence of an element y in A such that $\sup A - \varepsilon < y$ so we are done (because $y \in A \subset A \cup B$). If $Z = \sup B$, the same argument works (replace A by B).

□

Exercise 5. Page 43, #1.

Proof. Let's show that $\sup S = 1$, where $S = \left\{1 - \frac{1}{n}, n \in \mathbb{N}\right\}$:

1. For any $x = 1 - \frac{1}{n}$, one has $x \leqslant 1$;
2. For any $\varepsilon > 0$, one can find an x in S such that $1 - \varepsilon < x \leqslant 1$: indeed by the archimedean property one knows the existence of an integer $n_\varepsilon > 1/\varepsilon$. Therefore $1/n_\varepsilon < \varepsilon$ and $1 - \varepsilon < 1 - 1/n_\varepsilon$.

□

Exercise 6. Page 43, #14.

Proof. As in the textbook, let $S := \left\{s \in \mathbb{R}; 0 \leqslant s \,\text{and}\, s^2 < 3\right\}$.

S is not empty (it contains 1 for example) and is bounded above (for example by 2, because $s \geqslant 2$ implies that $s^2 \geqslant 4$ and thus such an s is not in S). Therefore, by completeness of $\mathbb{R}$ we know the existence of $x = \sup(S)$. Let's prove now that $x^2 = 3$.

1. **$x^2 > 3$ is impossible:**

   It is enough to find an integer $n \geqslant 1$ such that $(x - \frac{1}{n})^2 > 3$. Because then any $s \in S$ would be such that $s^2 < (x - \frac{1}{n})^2$, implying $s < (x - \frac{1}{n})$ (because $s \geqslant 0$ and $(x - \frac{1}{n}) > 0$); but that last inequality would mean that $(x - \frac{1}{n})$ is an upper bound of S (absurd).

   Let's find such an integer n:

   we notice that $(x - \frac{1}{n})^2 = x^2 - \frac{2.x}{n} + \frac{1}{n^2} \geqslant x^2 - \frac{2.x}{n}$. So we would be done if we could find n such that $x^2 - \frac{2.x}{n} > 3$, but this is equivalent to finding an n such that $\frac{x^2 - 3}{2.x} > \frac{1}{n}$ where x is given to you. But we know that this is possible, by the archimedean property of $\mathbb{R}$.

   Remark: other possible proof:

   Remark that $x^2 > 3$ implies $x > 3/x$, therefore one has that $y = \frac{1}{2}(x + \frac{3}{x}) < x$. But now $y^2 > 3$. Indeed $y^2 - 3 = \frac{1}{4}(x^2 + 6 + \frac{9}{x^2} - 12) = \left[\frac{1}{2}(x - \frac{3}{x})\right]^2 > 0$.
2. **$x^2 < 3$ is impossible:**

   If one can finds an integer n such that $(x + \frac{1}{n})^2 \leqslant 3$, we are done (because we found an element of S strictly larger than $\sup S$,which is absurd).

   Notice that $(x + \frac{1}{n})^2 = x^2 + \frac{2.x}{n} + \frac{1}{n^2} \leqslant x^2 + \frac{2.x}{n} + \frac{1}{n}$. So we will be done if we can find an integer n such that $x^2 + \frac{2.x}{n} + \frac{1}{n} \leqslant 3$, which is equivalent to $\frac{1}{n}(2.x + 1) \leqslant 3 - x^2$, or if one prefers $n \geqslant \frac{2x+1}{3-x^2}$ (notice that $3 - x^2 \neq 0$, so I can divide by it!). But such an integer can always be found, given x, thanks to the archimedean property of $\mathbb{R}$.

□

Exercise 7. Page 43, #18.

Proof. Since $u > 0$, we know that $x < y$ implies $x/u < y/u$. Then we know the existence of a rational number $r \in \mathbb{Q}$ such that $x/u < r < y/u$. But this implies that $x < r.u < y$.

□