Correction of HW1

Exercise 1. Page 15, #2.

Proof. By induction:

1. The property is true for $n=1$, because $1^3=\left[\frac{1}{2}.1.2\right]^2$
2. Assume that the property is true for n, and prove that it's true for $n+1$:

$$\begin{aligned} 1^3+\ldots+n^3 &= \left[\frac{1}{2}.n.(n+1)\right]^2 \\ \Rightarrow\ 1^3+\ldots+n^3+(n+1)^3 &= \left[\frac{1}{2}.n.(n+1)\right]^2+(n+1)^3 \\ \Rightarrow\ 1^3+\ldots+(n+1)^3 &= \frac{1}{4}.(n+1)^2\left[n^2+4(n+1)\right] \\ \Rightarrow\ 1^3+\ldots+(n+1)^3 &= \frac{1}{4}.(n+1)^2[n+2]^2 \\ \Rightarrow\ 1^3+\ldots+(n+1)^3 &= \left[\frac{1}{2}.(n+1).(n+2)\right]^2 \end{aligned}$$

But this is exactly the property for (n+1).

□

Exercise 2. Page 29, #3.

Proof. a) $2x+5=8 \ \Rightarrow 2x=3$(existence of negative elements) $\Rightarrow x=3/2$(existence of inverse for nonzero elements).

b) add $-2x$ to both sides to get $x^2-2x=0$. Then factor (using distributivity) to get $x(x-2)=0$. Conclude with theorem 2.1.3.

c) add -3 to both sides to get $x^2-4=0$, use distributivity to factor and conclude like in b).

d) Same: apply theorem 2.1.3 ($a.b=0$ implies $a=0$ or $b=0$).

□

Exercise 3. Page 30, #8.

Proof. a) Clearly $\frac{a}{b}+\frac{c}{d}=\frac{a.d+b.c}{b.d}\in\mathbb{Q}$ and $\frac{a}{b}.\frac{c}{d}=\frac{a.c}{b.d}\in\mathbb{Q}$.

b) If x is rational and y irrational, then $x+y$ can't be rational because $y=(x+y)+(-x)$ would also be rational.

Now if in addition $x\neq 0$, then $x.y$ can't be rational because $y=(x.y).(1/x)$ would also be rational from a).

□

Exercise 4. Page 30, #18.

Proof. By contradiction: assume that $a > b$. Then take $\varepsilon = \frac{a-b}{2}$. One should have $a - b \leqslant (\frac{a-b}{2})$ which is absurd. □

Exercise 5. Page 30, #23.

Proof. Let us prove that if $a, b > 0$ then $a < b$ if and only if $a^n < b^n$ for any $n \in \mathbb{N}$.

Clearly the right hand side implies the left one.

Let's prove the rest by induction:

1. The property is true for $n = 1$ because $a < b \Rightarrow a^1 < b^1$.
2. Assume that one has $a^n < b^n$. Then $a.a^n < a.b^n < b.b^n$ so we are done.

□

Exercise 6. Page 34, #1.

Proof. a) Clearly $|a|^2 = a^2$, and since $|a| \geq 0$, it is the square root of a^2.

b) Just notice that $|a| = |\frac{a}{b}.b| = |\frac{a}{b}|.|b|$ which gives the result.

□

Exercise 7. Page 34, #6.

Proof. a) $|4.x - 5| \leqslant 13$ is equivalent to

$$\begin{gathered} -13 \leqslant 4.x - 5 \leqslant 13 \\ -13 + 5 \leqslant 4.x \leqslant 13 + 5 \\ -2 \leqslant x \leqslant 9/2 \end{gathered}$$

so this is equivalent to $x \in [-2, 9/2]$.

b) $|x^2 - 1| \leqslant 3$ is equivalent to

$$\begin{gathered} -3 \leqslant x^2 - 1 \leqslant 3 \\ -2 \leqslant x^2 \leqslant 4 \end{gathered}$$

but the last line is equivalent to $0 \leqslant x^2 \leqslant 4$, which is itself equivalent to $x \in [-2, 2]$.

□

Exercise 8. Page 34, #15.

Proof. Assume $a < b$, then any positive real number strictly less than $(b-a)/2$ will work.

Take $\varepsilon = (b-a)/2$, then $U = (a - \frac{b-a}{2}, \frac{a+b}{2})$ and $V = (\frac{a+b}{2}, b + \frac{b-a}{2})$, and these two open intervals are disjoint.

It's probably cleaner to use $\varepsilon = \frac{b-a}{3}$ instead...

□