MAT 319 Spring 2015
Notes on power series

A *power series* is a series $\sum_0^\infty a_k x^k$. The main questions are:

- for which values of x is the series convergent?
- for those values, what are the properties of the function $f(x) = \sum_0^\infty a_k x^k$ defined by the series?

Proposition 1: Suppose that the sequence $|\dfrac{a_{n+1}}{a_n}|$ has a limit β as $n \to \infty$; β may be $0, \infty$ or any number in between. Let $R = 1/\beta$. Then $\sum_0^\infty a_k x^k$ converges for $|x| < R$ and diverges for $|x| > R$. R is the *radius of convergence* of the power series.

Proof: Apply the Ratio Test to $\sum_0^\infty a_k x^k$. The ratio of consecutive terms is $|a_{n+1}x^{n+1}/a_n x^n| = |x||\frac{a_{n+1}}{a_n}|$. The limit of these ratios is $|x|\beta = |x|/R$ which will be < 1 if $|x| < R$ and > 1 if $|x| > R$.

Note that since the Ratio Test gives no information when the limit of the ratios is 1, Proposition 1 only yields the *radius of convergence* R, but not what happens at $x = R$ or $x = -R$. These have to be analyzed separately as series. As we saw in class,

- $\displaystyle\sum_{x=0}^{\infty} x^n$ has interval of convergence $(-1, 1)$
- $\displaystyle\sum_{x=1}^{\infty} \frac{x^n}{n}$ has interval of convergence $[-1, 1)$
- $\displaystyle\sum_{x=1}^{\infty} \frac{x^n}{n^2}$ has interval of convergence $[-1, 1]$.

If a number belongs to the interval of convergence of the series $\sum_{k=0}^\infty a_k x^k$, that means that the series converges to a certain value when that number is substituted for x; this defines a function f which we write as $f(x) = \sum_{k=0}^\infty a_k x^k$. By the usual definition of a series, $f(x) = \sum_{k=0}^\infty a_k x^k = \lim_{n\to\infty} \sum_{k=0}^n a_k x^k$. Now $g_n(x) = \sum_{k=0}^n a_k x^k = a_0 + a_1 x + \cdots + a_n x^n$ defines an ordinary polynomial of degree n. In particular g_n is continuous (and has derivatives of any order). But examples like $g_n(x) = x^n$ for $0 \leq x \leq 1$, where $\lim_{n\to\infty} g_n(x) = 0$ if $x < 0$ and $= 1$ if $x = 1$ show that the limit of a sequence of continuous functions may be discontinuous. A stronger kind of limit is needed.

Definition: We are given a sequence (g_n) of functions defined on a common domain S. The sequence (g_n) *converges uniformly* to the function f defined on S if for every $\epsilon > 0$ there exists an index N with the property that $n > N$ implies $|g_n(x) - f(x)| < \epsilon$ for every $x \in S$. (Ross, Def. 24.2).

Equivalently, we say "f is the *uniform limit* of the sequence (g_n)."

It is important to understand the difference between this limit and the one implied in the statement $f(x) = \sum_{k=0}^{\infty} a_k x^k$. There for any $x \in S$ we know that $f(x) = \lim_{n\to\infty} g_n(x)$, with g_n as above. (We say that "f is the *pointwise limit* of the sequence (g_n)"). This means that for any $x \in S$, for every $\epsilon > 0$ there exists an index N with the property that $n > N$ implies $|g_n(x) - f(x)| < \epsilon$. *But here N may depend on x!*. Everything depends on the placement of the clause "for any $x \in S$."

Proposition 2: A uniform limit of continuous functions is continuous. (Ross, Theorem 24.3).

Our immediate goal is to prove that the function defined by a convergent power series $\sum_{k=0}^{\infty} a_k x^k$ is continuous within its radius of convergence, so it would be useful to know where the sequence (g_n) of functions, defined by $g_n(x) = \sum_{k=0}^{n} a_k x^k$, is uniformly convergent. To settle this we need an additional concept, the adaptation of "Cauchy sequence" to uniform convergence of a sequence of functions.

Definition: A sequence (g_n) of functions defined on a common domain S is *uniformly Cauchy* if for every $\epsilon > 0$ there exists an index N with the property that if $n, m > N$ then $|g_n(x) - g_m(x)| < \epsilon$ for any $x \in S$.

Note again the placement of the clause "for any $x \in S$."

Proposition 3. If a sequence (g_n) of functions defined on a common domain S is uniformly Cauchy, then it converges uniformly to a function f defined on S. (Ross, Theorem 25.4).

A tool for applying Proposition 3 to power series is the "Weierstrass M-test."

Proposition 4. (Ross 25.7) We start with a convergent series $\sum_{k=0}^{\infty} M_k = L$ of positive numbers. Now suppse we have an infinite series $\sum_{k=0}^{\infty} f_k(x)$ of functions all defined on some domain S, and that for eack k, $|f_k(x)| \leq M_k$ for all $x \in S$. Then the sequence g_n of functions, defined by $g_n(x) = \sum_{k=0}^{n} f_k(x)$, converges uniformly on S to a limit $f(x)$. Or we can say that the series $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly on S to f.

The proof uses the logic $\sum_{k=0}^{\infty} M_k$ convergent means $(\mathbf{M}_n) = (\sum_{k=0}^{n} M_k)$ is a convergent sequence $\Rightarrow$ $(\mathbf{M}_n)$ is a Cauchy sequence $\Rightarrow$ the sequence $(g_n(x))$ is uniformly Cauchy for $x \in S$, and then Proposition 3.

Now we are in a position to prove that a power series defines a *continuous* function inside its radius of convergence.

Lemma 1. Suppose that for a power series $\sum_{k=0}^{\infty} a_k x^k$ the limit $\lim_{k\to\infty} |\dfrac{a_{k+1}}{a_k}| = \beta$ exists (β can be 0, ∞ or anything in between), so that $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R = 1/\beta$. The associated power series $\sum_{k=0}^{\infty} |a_k| x^k$ (each coefficient is replaced by its absolute value) also has radius of convergence R.

Proof: Follows from $|\dfrac{|a_{k+1}|}{|a_k|}| = |\dfrac{a_{k+1}}{a_k}|$.

Proposition 5. (Ross, 26.1) A power series $\sum_{k=0}^{\infty} a_k x^k$ with radius of convergence $0 < R \leq \infty$ converges uniformly on any interval $[-R_1, R_1]$ with $R_1 < R$.

Proof: Let $M_k = |a_k| R_1^k$. Since $R_1 < R$, it is inside the radius of convergence of $\sum_{k=0}^{n} |a_k| x^k$ (see Lemma 1), so $\sum_{k=0}^{\infty} M_k$ is a convergent series of positive numbers. Furthermore the functions $f_k = a_k x^k$ satisfy $|f_k(x)| \leq M_k$ for $|x| \leq R_1$. By the Weierstrass M-test, $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly to a function $f(x)$ on $[-R_1, R_1]$.

Note that since each f_k is continuous and the convergence is uniform, the limit function f is continuous on $[-R_1, R_1]$.

Proposition 6. (Ross 26.2) If the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence R, then the sum $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is continuous on $(-R, R)$.

Proof: It is enough to show that f is continuous at any $x \in (-R, R)$. Since $|x| < R$, there exists R_1 with $|x| < R_1 < R$. Since the series converges uniformly on $[-R_1, R_1]$ by Proposition 5, the limit is continuous on $[-R_1, R_1]$, hence at x.

Note that even though the limit function is continuous on $(-R, R)$, the convergence is not, in general, uniform on $(-R, R)$. As $R_1 \to R$, the uniform convergence given by Proposition 5 may become harder and harder to achieve, in the sense that larger and larger Ns are required for any given ϵ.

Integration and differentiation of power series.

Lemma 2. Suppose that for a power series $\sum_{k=0}^{\infty} a_k x^k$ the limit $\lim_{k\to\infty} |\frac{a_{k+1}}{a_k}| = \beta$ exists (β can be 0, ∞ or anything in between), so that $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R = 1/\beta$. Then the associated power series $\sum_{k=0}^{\infty} k a_k x^{k-1}$ and $\sum_{k=0}^{\infty} \frac{1}{k+1} a_k x^{k+1}$ also have radius of convergence R.

Proof: To follow the regular construction explicitly, let $b_i = (i+1)a_{i+1}$ and $c_j = \frac{1}{j} a_{j-1}$, so that

$$\sum_{k=0}^{\infty} k a_k x^{k-1} = \sum_{i=0}^{\infty} b_i x^i \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{k+1} a_k x^{k+1} = \sum_{j=1}^{\infty} c_j x^j.$$

Then $\dfrac{b_{i+1}}{b_i} = \dfrac{(i+2)a_{i+2}}{(i+1)a_{i+1}} = \dfrac{i+2}{i+1}\dfrac{a_{i+2}}{a_{i+1}}$. Since $\lim_{i\to\infty} \dfrac{i+2}{i+1} = 1$ and $\lim_{i\to\infty} \dfrac{a_{i+2}}{a_{i+1}} = \beta$ the first associated power series also has radius of convergence $R = 1/\beta$. On the other hand $\dfrac{c_{j+1}}{c_j} = \dfrac{\frac{1}{j+1}a_j}{\frac{1}{j}a_{j-1}} = \dfrac{j}{j+1}\dfrac{a_j}{a_{j-1}}$. Since $\lim_{j\to\infty} \dfrac{j}{j+1} = 1$ and $\lim_{j\to\infty} \dfrac{a_j}{a_{j-1}} = \beta$ the second associated power series also has radius of convergence R.

Proposition 7. (Ross, 25.2) Let (f_n) be a sequence of continuous functions defined on $[a, b]$ which converges uniformly on $[a, b]$ to the function f. Then $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$.

Note that proof requires uniform convergence.

Proposition 8. (Ross, 26.4) If the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of convergence

$R > 0$, then for any x with $|x| < R$,

$$\int_0^x f(t)\ dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^k.$$

Proof: On the interval $[-x, x]$ the series converges uniformly (Proposition 5); this means that the sequence of partial sums $g_n(x) = \sum_{k=0}^{n} a_k x^k$ converges uniformly to f; so by Proposition 7,

$$\int_0^x f(t)\ dt = \lim_{n\to\infty} \int_0^x g_n(t)\ dt = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1} = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}.$$

This proposition means that a uniformly convergent power series can be integrated term by term.

Proposition 9. (Ross, 26.5) If the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R > 0$, then f is differentiable on $(-R, R)$ and for $|x| < R$, $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$.

Proof: By Lemma 2, the series $g(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}$ also has radius of convergence R, so we can apply Proposition 8 and integrate it term by term from 0 to x:

$$\int_0^x g(t)\ dt = \sum_{k=1}^{\infty} \frac{1}{k-1+1} k a_k x^{k-1+1} = \sum_{k=1}^{\infty} a_k x^k = f(x) - a_0.$$

Now differentiate both sides. By the Fundamental Theorem of Calculus, $(d/dx) \int_0^x g(t)\ dt = g(x)$, whereas $(d/dx)(f(x) - a_0) = f'(x)$.

Some important examples.

We are now in a position to prove (with a little bit of calculus) that the three series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$g(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$h(x) = \sum_{k=1}^{\infty} (-1)^{k} \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$$

are equal to e^x, $\sin x$ and $\cos x$, respectively.

Note that all three series have radius of convergence $R = \infty$, so the functions f, g, h are continuous on $(-\infty, \infty)$. Furthermore by Proposition 9, each of them is differentiable with derivative given by another power series with $R = \infty$, so the derivatives themselves are differentiable, and so on: each of f, g, h is infinitely differentiable on the whole line.

1. Differentiating f term by term gives back the same series. Since we know by Proposition 9 that this series represents f', the function f satisfies the differenial equation $f' = f$, with initial condition (as can be checked from the series) $f(0) = 1$. Now some calculus: we know from the Chain Rule that $f'/f = (\ln(f))'$, and so

$$\int_0^x \frac{f'(t)}{f(t)}\, dt = \ln(f(x)) - \ln(f(0)) = \ln(f(x)),$$

using the initial condition. On the other hand since $f'/f = 1$,

$$\int_0^x \frac{f'(t)}{f(t)}\, dt = \int_0^x 1\, dt = x.$$

So $\ln(f(x)) = x$. Exponentiating both sides gives $f(x) = e^x$.

2. Differentiating g term by term gives exactly the series h, and differentiating h term by term gives the series g with a minus sign. By Proposition 9, $h = g'$ and $-g = h' = g''$, so the function g satisfies the differenial equation $g'' = -g$, with initial conditions (as can be checked from the series) $g(0) = 0$ and $g'(0) = h(0) = 1$. This differential equation, with these initial conditions, is also satisfied by the function $\sin x$.

Proposition 10. Suppose $p(x)$ and $q(x)$ are defined for $\infty < x < \infty$ and are both solutions of the equation $y'' = -y$ with initial conditions $y(0) = 0, y'(0) = 1$. Then $p(x) = q(x)$ for all x.

Proof: Consider the difference $u = p - q$. It also satisfies $u'' = -u$, but with initial conditions $u(0) = 0, u'(0) = 0$. Writing $u' = v$ transforms the second-order equation $u'' = -u$ into the equivalent set of two coupled first-order equations $u' = v$ and $v' = -u$; the initial conditions are now $u(0) = 0, v(0) = 0$. Define a new function $E = u^2 + v^2$. Note that $E(0) = 0$. Calculating E' using the Chain Rule gives $E' = 2uu' + 2vv'$; substituting $u' = v$ and $v' = -u$ gives $E' = 2uv + 2v(-u) = 0$. From calculus we know that a function defined on $(-\infty, \infty)$ (or on any interval) with zero derivative must be constant. Therefore E must be constant, and since $E(0) = 0$, that constant value must be 0. Going back to the definition of E, this means $u^2 + v^2 = 0$, which is only possible if u and v are everywhere 0. In particular $u = 0$ so $p = q$ everywhere.

Proposition 10 implies that the function f defined by the power series, and the function sin, are the same function.

3. This argument could be repeated to prove that $h(x) = \cos x$. But it's enough to observe that $h(x)$ is the term by term derivative of the series $g(x)$, and so by Proposition 9 it is the derivative of the function defined by $g(x)$. Since we have just established that $g(x) = \sin x$, it follows that $h(x) = g'(x) = \cos x$.

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