**MAT 319 Spring 2015** **A natural example of a Cauchy Sequence.**

*Definition:* $(s_n)$ is a Cauchy sequence if for every $\epsilon > 0$ there exists a natural number $N$ such that if $m, n > N$ then $|s_m - s_n| < \epsilon$.

*Example:* Using the Fibonacci sequence $F_1 = 1, F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$, define $s_n = F_{n+1}/F_n$, so $s_1 = 1$, $s_2 = 2$, $s_3 = 3/2$ etc.

*Proposition:* In this example, $(s_n)$ is a Cauchy sequence.

Proof: First note that

$$|s_{n+1} - s_n| = |\frac{F_{n+2}}{F_{n+1}} - \frac{F_{n+1}}{F_n}| = |\frac{F_{n+2}F_n - F_{n+1}^2}{F_{n+1}F_n}| = |\frac{1}{F_{n+1}F_n}| \quad (*)$$

using a Fibonacci identity proved earlier.

Also note that supposing (as we may w.l.o.g) that $m > n$,

$$|s_m - s_n| = |(s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \cdots + (s_{n+2} - s_{n+1}) + (s_{n+1} - s_n)|$$

which we may rewrite as

$$|s_m - s_n| = |(s_{n+1} - s_n) + (s_{n+2} - s_{n+1}) + \cdots + (s_{m-1} - s_{m-2}) + (s_m - s_{m-1})|.$$

By the triangle inequality,

$$|s_m - s_n| \leq |s_{n+1} - s_n| + |s_{n+2} - s_{n+1}| + \cdots + |s_{m-1} - s_{m-2}| + |s_m - s_{m-1}|.$$

Using $(*)$, this becomes (we can leave out the $||$ because the terms are positive)

$$|s_m - s_n| \leq \frac{1}{F_{n+1}F_n} + \frac{1}{F_{n+2}F_{n+1}} + \cdots + \frac{1}{F_{m-1}F_{m-2}} + \frac{1}{F_mF_{m-1}}$$

which we can write in compressed form as

$$|s_m - s_n| \leq \sum_{k=n}^{m-1} \frac{1}{F_{k+1}F_k} < \sum_{k=n}^{\infty} \frac{1}{F_{k+1}F_k}$$

since we can add extra positive terms to the right-hand side. [This is now different from the attempt in class]. The terms $\frac{1}{F_{k+1}F_k}$ go to zero very rapidly, but for our purposes it is enough to remark that $F_k > k$ as soon as $k \geq 5$ so

$$|s_m - s_n| < \sum_{k=n}^{\infty} \frac{1}{k(k+1)} < \sum_{k=n}^{\infty} \frac{1}{k^2} < \int_{n-1}^{\infty} \frac{1}{x^2}\, dx = \frac{1}{n-1}$$

using the "integral test." Now we know that $|s_m - s_n| < \frac{1}{n-1}$ where $n$ is the smaller of the two indices. So if $N \geq 1/\epsilon + 1$ then $m, n > N$ implies

$$|s_m - s_n| < \frac{1}{n-1} < \frac{1}{N-1} \leq \epsilon,$$

as required.