MAT 312 Spring 2009 Review for Midterm 2

2.1 Understand that the concept "symmetry" (loosely defined on p. 62) depends on the context. In particular the 24 symmetries of a regular tetrahedron (problem 12) cannot all be realized by physical motions in space since half of them involve a reflection. Understand that in composing symmetries the *order* is important: Table 2.1 is not symmetrical about the diagonal ($h_2 \circ h_5 = h_7, \quad h_5 \circ h_2 = h_8$); Problem 13.

2.2 Understand and be able to apply the definition of *group* and the discussion (top of p. 73) of the "closure axiom." Be able to tell whether or not a composition table is the composition table of a group (Example 2.13). Understand Theorem 2.1 (symmetries of a polygon form a group). Know what "commutative" means (Definition 2.4) and that the word "abelian" is often used for this property.

2.3 Be familiar with $\{\mathbf{Z}_n, +\}$ the additive group of integers modulo n, and with the construction of the *product* $(G \times H, \star)$ of groups $(G, \circ)$ and $(H, \cdot)$ (Definition 2.6). Be able to prove that $(G\times H, \star)$ satisfies the group axioms (Theorem 2.3). Understand the concept of *isomorphism* and be able to prove that $\mathbf{Z}_4$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$ are not isomorphic (Example 2.25).

2.4 Be able to prove Theorem 2.4 (uniqueness of identity) and Theorem 2.5 (uniqueness of inverses), and understand exactly where the group axioms enter into these proofs. Be able to prove that $(b \circ a)^{-1} = a^{-1} \circ b^{-1}$ (note order! *note change, 03/31*) and understand it in terms of common sense, e.g. $a =$ *open refrigerator* and $a^{-1} =$ *close refrigerator*, $b =$ *take out milk* and $b^{-1} =$ *put in milk*. Be able to prove that $(a^{-1})^{-1} = a$ (Theorem 2.7) and Theorem 2.8: in a group, the equation $a \circ x = b$ has a *unique* solution x.

2.5 Understand the concept of *subgroup* (Definition 2.7) and the content of Theorem 2.9 (if H is a subgroup of G, then the composition in H is the composition in G, applied to pairs of elements of H). Understand the distinction between a *finite* group (e.g. $(\mathbf{Z}_n, +)$) and an *infinite* one (e.g. $(\mathbf{Z}, +)$). Understand that the *order* of $g \in G$ is the smallest positive integer n with $g^n = e$, and be able to prove that in a finite group every element has finite order (Theorem 2.11). Know some examples of subgroups. The *cyclic subgroup* $< g >$ generated by $g \in G$ is important. Be able to prove Theorem 2.12. Understand why a nonempty subset S of a *finite* group is a subgroup if and only if it contains the composition of any two of its elements (not true if group not finite!).

2.6 Know the definition of S_n, the *symmetric group on n letters*. Understand cycle notation (pp. 114-115) and be able to represent any permutation as a product of *disjoint* cycles (Examples 2.62, 2.63, 2.64). Understand (Example 2.64) that in a product of cycles, the order of application of the permutations is right to left – like composition of functions.

3.1 The basic concept here is *equivalence relation* (Definition 3.3). The main example we study is G-equivalence: here G is a subgroup of the group of all permutations of a set X (we call this "a group of permutations of X") and for $x, y \in X$, $x \sim_G y$ if there exists $g \in G$ with $g(x) = y$. Understand how the 3 conditions (reflexive, symmetric, transitive) that make $\sim_G$ an equivalence relation follow from the three axioms (identity, inverses, associativity) for G (Proposition 3.1). In general, an equivalence relation on X partitions X into *disjoint* equivalence classes: $[x] = \{y \in X \mid y \sim_G x\}$; be able to prove Proposition 3.2. Know some examples like $X = \mathbf{Z}$ and $x \equiv y \mod n$ if $x - y$ is a multiple of n (Example 3.9); also Example 3.10 which is directly given as a G-equivalence.

3.2 Understand Proposition 3.4. It treats a special case of G-equivalence where now the set is G and the group is a subgroup H of G. For any $h \in H$, $g \to gh$ defines a permutation of the elements of G (understand this point!), so H can be considered as a subgroup of the group of all permutations of the elements of G. In this case the equivalence class of g is called the corresponding *left coset* and written $[g] = gH$; the notation is natural because everything in $[g]$ is of the form gh for some $h \in H$. Note that $H = eH$ is automatically a coset (top of p. 133). Understand Examples 3.16 and 3.17.

Be able to prove Proposition 3.5 which says essentially that all left cosets have the same number of elements, which is $|H|$, the *order* of H (Definition 3.6). This proposition has Lagrange's Theorem (Theorem 3.6) as an immediate consequence; understand how this works. Understand how this implies that the order of any $g \in G$ must divide $|G|$ (Theorem 3.7) and that therefore any group of prime order must be cyclic (Theorem 3.8).

3.3 Start with Definition 3.7 (G is a group of permutations of X): Know how to define the *fixed point set* X_g for $g \in G$ and the *stabilizer* G_x of an element $x \in X$. Be able to prove Proposition 3.9 (G_x is a subgroup of G). Proposition 3.10 ($|G_x| = |G|/|[x]|$ for any $x \in X$) is the main ingredient in the proof of Burnside's Theorem. Understand how the proof works. Understand also the proof of Burnside's Theorem (Theorem 3.11) itself: it follows from the counting argument $\sum_x |G_x| = \sum_g |X_g|$, Proposition 3.10, and the observation that if $x \sim_G y$ then $|[x]| = |[y]|$. The examples of applications of Burnside's Theorem are important: Examples 3.31, 3.34, 3.35.