

MAT 312/AMS 351

Applied Algebra

Midterm 2 – Solutions

1. (a) (10 points) Show that the groups $\mathbf{Z}_3 \times \mathbf{Z}_3$ and \mathbf{Z}_9 are not isomorphic. (Remember that two groups are isomorphic if one can be considered a re-labeling of the other: the same algebra with different tags).

SOLUTION In $\mathbf{Z}_3 \times \mathbf{Z}_3$, every element except 0 has order 3, since $(a, b) + (a, b) + (a, b) = (a + a + a, b + b + b)$ and $1 + 1 + 1 = 2 + 2 + 2 = 0$ in \mathbf{Z}_3 . In \mathbf{Z}_9 the element 1 has order 9. So the algebraic structures are different.

- (b) (15 points) Show that the groups $\mathbf{Z}_3 \times \mathbf{Z}_2$ and \mathbf{Z}_6 are isomorphic.

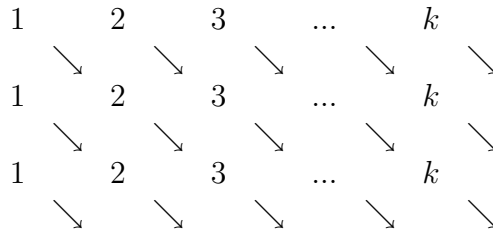
SOLUTION It is enough to show that $\mathbf{Z}_3 \times \mathbf{Z}_2$ is cyclic of order 6. We can use either $(1, 1)$ or $(2, 1)$ as generator:

$$\begin{aligned}
 &(1, 1) \\
 &(1, 1) + (1, 1) = (2, 0) \\
 &(1, 1) + (2, 0) = (0, 1) \\
 &(1, 1) + (0, 1) = (1, 0) \\
 &(1, 1) + (1, 0) = (2, 1) \\
 &(1, 1) + (2, 1) = (0, 0).
 \end{aligned}$$

The second version of the test had $\mathbf{Z}_2 \times \mathbf{Z}_3$.

2. (25 points) The group \mathbf{S}_n of all permutations of n distinct objects, say, 1, 2, 3, \dots , n has order $n!$. Show that for every k , $1 \leq k \leq n$, \mathbf{S}_n has a subgroup of order exactly k , by exhibiting such a subgroup.

SOLUTION The cyclic subgroup generated by (12) has 2 elements: $\{(12), (12)^2 = e\}$. The cyclic subgroup generated by (123) has 3 elements: $\{(123), (123)^2 = (132), (123)^3 = e\}$. Similarly the cyclic subgroup generated by any k -cycle, for example $(123\dots k)$ has order k :



(after k steps, each element is back where it started). So for each $k \leq n$ the cyclic subgroup $\langle (123\dots k) \rangle$ has order k .

3. (25 points) Two colorings of the vertices of an equilateral triangle are considered equivalent if a symmetry of the triangle takes one to the other. In how many *non-equivalent* ways can one color the vertices of an equilateral triangle with 5 colors?

SOLUTION We apply Burnside's Theorem. If the triangle has vertices A, B, C the group G of symmetries is $\{e, (AB), (AC), (BC), (ABC), (ACB)\}$. All possible distinct colorings are invariant under e : there are 5^3 of them, since each vertex can have any one of the 5 colors; so $|X_e| = 5^3$. The colorings invariant under (AB) must give A and B the same color; the color for C can be chosen independently, so there are 5^2 ways of doing this, and $|X_{(AB)}| = 5^2$. Same for $X_{(AC)}$ and $X_{(BC)}$. The colorings invariant under (ABC) must give all 3 vertices the same color; this can be done 5 ways. So $|X_{(ABC)}| = 5$ and same for $X_{(ACB)}$. Burnside's Theorem says the number of distinct colorings up to symmetry is

$$k = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

In our case this gives $(1/6)(5^3 + 3 \cdot 5^2 + 2 \cdot 5) = 35$.

The other test had 4 colors, with number of distinct colorings up to symmetry equal to $(1/6)(4^3 + 3 \cdot 4^2 + 2 \cdot 4) = 20$.

4. (25 points) The group \mathbf{S}_4 has $4! = 24$ elements:

$$\begin{aligned} &e \\ &(12), (13), (14), (23), (24), (34) \\ &(123), (132), (124), (142), (134), (143), (234), (243) \\ &(1234), (13)(24), (1432), (1324), (12)(34), (1423), (1243), (14)(23), (1342) \end{aligned}$$

The subgroup $H_2 = \{\text{all permutations that fix the element 2}\}$ has 6 elements:

$$e, (13), (14), (34), (134), (143).$$

What are the four left cosets determined by H_2 ?

SOLUTION The first coset is H_2 itself: $\{e, (13), (14), (34), (134), (143)\}$.

To get another coset take any element not in H_2 , e.g. (12) , and calculate $(12)H_2 = \{(12), (132), (142), (12)(34), (1342), (1432)\}$.

For a third coset, take any element not in H_2 or in $(12)H_2$, e.g. (23) , and calculate $(23)H_2 = \{(23), (123), (23)(14), (234), (1234), (1423)\}$.

For the last coset you can take what's left over, or take an element not in any of the first 3, say (24) , and use $(24)H_2 = \{(24), (24)(13), (124), (243), (1324), (1243)\}$.

The other test used H_3 , similarly defined, and the cosets were:

$$\begin{aligned}H_3 &= \{e, (12), (14), (24), (124), (142)\}, \\(13)H_3 &= \{(13), (123), (143), (13)(24), (1243), (1423)\}, \\(23)H_3 &= \{(23), (132), (23)(14), (243), (1324), (1432)\}, \\(34)H_3 &= \{(34), (34)(12), (134), (234), (1234), (1342)\}.\end{aligned}$$