MAT 312/AMS 351
Applied Algebra
Midterm 2 – Solutions

1. (a) (10 points) Show that the groups $\mathbf{Z}_3 \times \mathbf{Z}_3$ and $\mathbf{Z}_9$ are not isomorphic. (Remember that two groups are isomorphic if one can be considered a re-labeling of the other: the same algebra with different tags).

   Solution In $\mathbf{Z}_3 \times \mathbf{Z}_3$, every element except 0 has order 3, since $(a,b) + (a,b) + (a,b) = (a+a+a, b+b+b)$ and $1+1+1 = 2+2+2 = 0$ in $\mathbf{Z}_3$. In $\mathbf{Z}_9$ the element 1 has order 9. So the algebraic structures are different.

   (b) (15 points) Show that the groups $\mathbf{Z}_3 \times \mathbf{Z}_2$ and $\mathbf{Z}_6$ *are* isomorphic.

   Solution It is enough to show that $\mathbf{Z}_3 \times \mathbf{Z}_2$ is cyclic of order 6. We can use either $(1,1)$ or $(2,1)$ as generator:

$$\begin{array}{l}
(1,1) \\
(1,1) + (1,1) = (2,0) \\
(1,1) + (2,0) = (0,1) \\
(1,1) + (0,1) = (1,0) \\
(1,1) + (1,0) = (2,1) \\
(1,1) + (2,1) = (0,0).
\end{array}$$

   The second version of the test had $\mathbf{Z}_2 \times \mathbf{Z}_3$.

2. (25 points) The group $\mathbf{S}_n$ of all permutations of n distinct objects, say, 1, 2, 3, $\ldots n$ has order $n!$. Show that for every k, $1 \leq k \leq n$, $\mathbf{S}_n$ has a subgroup of order exactly k, by exhibiting such a subgroup.

   Solution The cyclic subgroup generated by (12) has 2 elements: $\{(12), (12)^2 = e\}$. The cyclic subgroup generated by (123) has 3 elements: $\{(123), (123)^2 = (132), (123)^3 = e\}$. Similarly the cyclic subgroup generated by any k-cycle, for example $(123...k)$ has order k:

$$\begin{array}{ccccccccccc}
1 & & 2 & & 3 & & ... & & k & & \\
 & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\
1 & & 2 & & 3 & & ... & & k & & \\
 & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\
1 & & 2 & & 3 & & ... & & k & & \\
 & \searrow & & \searrow & & \searrow & & \searrow & & \searrow &
\end{array}$$

   (after k steps, each element is back where it started). So for each $k \leq n$ the cyclic subgroup $< (123...k) >$ has order k.

3. (25 points) Two colorings of the vertices of an equilateral triangle are considered equivalent if a symmetry of the triangle takes one to the other. In how many *non-equivalent* ways can one color the vertices of an equilateral triangle with 5 colors?

SOLUTION We apply Burnside's Theorem. If the triangle has vertices A, B, C the group G of symmetries is $\{e, (AB), (AC), (BC), (ABC), (ACB)\}$. All possible distinct colorings are invariant under e: there are 5^3 of them, since each vertex can have any one of the 5 colors; so $|X_e| = 5^3$. The colorings invariant under (AB) must give A and B the same color; the color for C can be chosen independently, so there are 5^2 ways of doing this, and $|X_{(AB)}| = 5^2$. Same for $X_{(AC)}$ and $X_{(BC)}$. The colorings invariant under (ABC) must give all 3 vertices the same color; this can be done 5 ways. So $|X_{(ABC)}| = 5$ and same for $X_{(ACB)}$. Burnside's Theorem says the number of distinct colorings up to symmetry is

$$k = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

In our case this gives $(1/6)(5^3 + 3 \cdot 5^2 + 2 \cdot 5) = 35$.

The other test had 4 colors, with number of distinct colorings up to symmetry equal to $(1/6)(4^3 + 3 \cdot 4^2 + 2 \cdot 4) = 20$.

4. (25 points) The group $\mathbf{S}_4$ has $4! = 24$ elements:

$$\begin{array}{l} e \\ (12), (13), (14), (23), (24), (34) \\ (123), (132), (124), (142), (134), (143), (234), (243) \\ (1234), (13)(24), (1432), (1324), (12)(34), (1423), (1243), (14)(23), (1342) \end{array}.$$

The subgroup $H_2 = \{\text{all permutations that fix the element } 2\}$ has 6 elements:

$$e, (13), (14), (34), (134), (143).$$

What are the four left cosets determined by H_2?

SOLUTION The first coset is H_2 itself: $\{e, (13), (14), (34), (134), (143)\}$.

To get another coset take any element not in H_2, e.g. (12), and calculate $(12)H_2 = \{(12), (132), (142), (12)(34), (1342), (1432)\}$.

For a third coset, take any element not in H_2 or in $(12)H_2$, e.g. (23), and calculate $(23)H_2 = \{(23), (123), (23)(14), (234), (1234), (1423)\}$.

For the last coset you can take what's left over, or take an element not in any of the first 3, say (24), and use $(24)H_2 = \{(24), (24)(13), (124), (243), (1324), (1243)\}$.

The other test used H_3, similarly defined, and the cosets were:

$$
\begin{aligned}
H_3 &= \{e, (12), (14), (24), (124), (142)\},\\
(13)H_3 &= \{(13), (123), (143), (13)(24), (1243), (1423)\},\\
(23)H_3 &= \{(23), (132), (23)(14), (243), (1324), (1432)\},\\
(34)H_3 &= \{(34), (34)(12), (134), (234), (1234), (1342)\}.
\end{aligned}
$$