MAT 312/AMS 351
Notes and exercises on normal subgroups and quotient groups.

If H is a subgroup of G, the equivalence relation $\sim_H$ is defined between elements of G as follows:

$$g_1 \sim_H g_2 \Leftrightarrow \exists h \in H, g_1 = g_2 h.$$

Proposition 1. This is indeed an equivalence relation.

Proof: The three properties: reflexive, symmetric, transitive correspond to the three properties of a subgroup: H contains the identity element e of G, H contains inverses of all its elements, H is closed under composition.

- For any $g \in G$, since $e \in H$ and $ge = g$, it follows that $g \sim_H ge = g$, so the relation $\sim_H$ is reflexive.
- If $g_1 \sim_H g_2$, $\exists h \in H, g_1 = g_2 h$. Since h^{-1} must then also belong to H, and $g_1 h^{-1} = g_2 h h^{-1} = g_2$, it follows that $g_2 \sim_H g_1$, so the relation $\sim_H$ is symmetric.
- If $g_1 \sim_H g_2$ and $g_2 \sim_H g_3$, then $\exists h \in H, g_1 = g_2 h$, and also $\exists h' \in H, g_2 = g_3 h'$. Since then $hh' \in H$, and $g_1 = g_2 h = (g_3 h')h = g_3(h'h)$, it follows that $g_1 \sim_H g_3$; so the relation $\sim_H$ is transitive.

For the $\sim_H$ equivalence class of the element $g \in G$ we have the suggestive notation gH (since every element of that equivalence class is gh for some $h \in H$); This equivalence class is called the *left H-coset* of g; "left" because gH is obtained by multiplying every element of H on the left by g. Note that the left H-coset of the identity e is H itself.

Right H-cosets. In a completely analogous way one can define $g_1 \equiv_H g_2 \Leftrightarrow \exists h \in H, g_1 = hg_2$. A completely analogous argument proves that, since H is a subgroup, the relation $\equiv_H$ is also an equivalence relation. In this case the equivalence class of $g \in G$ is written Hg and called the *right H-coset* of g.

Example. Consider $G = S(3)$, the group of permutations of 3 elements, so in cycle notation $G = \{e, (12), (13), (23), (123), (132)\}$; and consider the subgroup $H = \{e, (12)\}$. Since $|G| = 6$ and $|H| = 2$, we expect 3 left H-cosets. They are

- $H = \{e, (12)\}$
- $(13)H = \{(13), (13)(12) = (123)\}$

- $(23)H = \{(23), (23)(12) = (132)\}$.

Note that $(123)H = (12)H$ and $(132)H = (23)H$. A coset can have several names!

On the other hand the three right H-cosets are

- $H = \{e, (12)\}$
- $H(13) = \{(13), (12)(13) = (132)\}$
- $H(23) = \{(23), (12)(23) = (123)\}$.

So in general left H-cosets and right H-cosets give two different partitions of G. But sometimes the partitions coincide. In this case H is called a *normal* subgroup of G. More formally:

Definition: The subgroup H of group G is called *normal* if $gH = Hg$ for every $g \in G$.

Example 1. With $G = S(3)$ as above, consider the subgroup $H = \{e, (123), (132)\}$. Since $|G| = 6$ and $|H| = 3$, we expect two left H-cosets and two right H-cosets. In either case one coset must be H itself; so the other one must contain the three remaining elements, namely $\{(12), (13), (23)\}$. So H is normal in G. The same thing will happen whenever $|H| = \frac{1}{2}|G|$; in this case we say that H is a subgroup of index 2, and we can state the proposition: Every subgroup of index 2 is normal.

Example 2. Consider $G = A(4)$, the group of all even permutations of 4 elements. In cycle notation,

$$G = \{e, (123), (124), (132), (134), (142), (143),$$

$$(234), (243), (12)(34), (13)(24), (14)(23)\}.$$

(Since exactly half the permutations in $S(4)$ are even, this G is an index-2 subgroup of $S(4)$, and hence a normal subgroup of $S(4)$). In G, consider the subgroup $H = \{e, (12)(34), (13)(24), (14)(23)\}$. Here with $|G| = 12, |H| = 4$ we expect three cosets.

The left cosets are

- $H = \{e, (12)(34), (13)(24), (14)(23)\}$
- $(123)H = \{(123), (123)(12)(34) = (134),$
  $(123)(13)(24) = (243), (123)(14)(23) = (142)\}$
- $(124)H = \{(124), (124)(12)(34) = (143),$
  $(124)(13)(24) = (132), (124)(14)(23) = (234])\}$

The right cosets are

- $H = \{e, (12)(34), (13)(24), (14)(23)\}$

- $H(123) = \{(123), (12)(34)(123) = (243),$
  $(13)(24)(123) = (142), (14)(23)(123) = (134)\}$
- $H(124) = \{(124), (12)(34)(124) = (234),$
  $(13)(24)(124) = (143), (14)(23)(124) = (132)\}$

Note that $(123)H = H(123)$ and $(124)H = H(124)$. So H is a normal subgroup of $A(4)$.

Quotient groups. When H is a normal subgroup of G, the law of composition of G induces a composition between H-cosets which makes this set also into a group. This is called the *quotient group* of G by H, and written G/H.

Definition: For two cosets gH and $g'H$ (H is a normal subgroup of G, but we write them as left cosets for explicitness) we define $gH \cdot g'H$ to be $(gg')H$.

Proposition 2. This operation is well-defined, and makes the set of cosets into a group.

Proof: First, we need to show the operation is *well-defined* because the result might be different if we had chosen different names (i.e. different representative elements) for the cosets gH and $g'H$. So suppose in fact that $\gamma \in gH$ and $\gamma' \in g'H$. We need to show that $\gamma\gamma'H = gg'H$. What we know is that there is an element $h \in H$ such that $\gamma = gh$, and an element h' such that $\gamma' = g'h'$. So $\gamma\gamma'H = (gh)(g'h')H$. Now $h'H = H$ since $h' \in H$, and $g'h'H = g'H = Hg'$ since H is normal. Furthermore $hH = H$ since $h \in H$, so $hg'h'H = hHg' = Hg'$, and finally $\gamma\gamma'H = gHg' = gg'H$ using normality again.

Next we need to show that this operation satisfies the three conditions required of a group law.

- Associativity. $g_1H \cdot (g_2H \cdot g_3H) = g_1H \cdot (g_2g_3)H = g_1(g_2g_3)H = (g_1g_2)g_3H = (g_1g_2)H \cdot g_3H = (g_1H \cdot g_2H) \cdot g_3H$.
- Identity. The coset $eH = H$ is the identity, since $eH \cdot gH = (eg)H = gH$, and $gH \cdot eH = (ge)H = gH$.
- Inverses. The inverse of gH is $g^{-1}H$, since $gH \cdot g^{-1}H = (gg^{-1})H = eH$ and $g^{-1}H \cdot gH = (g^{-1}g)H = eH$.

Example: With $G = A(4)$ and H as above, $G/H = \{H, (123)H, (124)H\}$. Since $|G/H| = 3$, a prime, the group G/H must be isomorphic to $\mathbf{Z}_3$.

In fact the composition table for G/H is

	H	$(123)H$	$(124)H$
H	H	$(123)H$	$(124)H$
$(123)H$	$(123)H$	$(132)H = (124)H$	$(13)(24)H = H$
$(124)H$	$(124)H$	$(14)(23)H = H$	$(142)H = (123)H$

,

which is a relabeling of

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

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Exercises.

(1) "A completely analogous argument proves that, since H is a subgroup, the relation $\equiv_H$ is also an equivalence relation." Write out the details of this argument.

(2) In $G = \mathbf{Z}_{21}^* = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$ show that the set $H = \{1, 4, 16\}$ is a subgroup. Since G is abelian, H is automatically normal. Identify the four elements of G/H and construct their multiplication table. Is this group isomorphic to $\mathbf{Z}_4$?

(3) Let H be a subgroup of G. Show that H is normal if and only if $ghg^{-1} \in H$ for every $g \in G, h \in H$. Another way of writing this is: $gHg^{-1} = H$ for every $g \in G$.