MAT 310 Spring 2008 Review for Midterm 1

Chapter 1 **Fundamental concepts:** $\mathbf{F}$ is a field; V is a vector space over $\mathbf{F}$; U is a subspace of V; the sum "+" and direct sum "$\oplus$" of two subspaces of V.

Know the standard examples of fields: $\mathbf{Q}, \mathbf{R}, \mathbf{C}$ and $\mathbf{Z}_p$ for p a prime number. Understand why $\mathbf{Z}$ is not a field and why $\mathbf{Z}_4$ is not a field.

Be able to prove the basic facts about vector spaces given in Propositions 1.2, 1.3, 1.4, 1.5, 1.6.

Subspaces: understand why $\{(x, y, z) | x+2y-z = 1\}$ is not a subspace of $\mathbf{R}^3$. *Exercise 5.* Be able to prove that the intersection of two subspaces is a subspace *Exercise 6* but their union is not, in general *Exercise 9.* Understand the difference between sum and direct sum (Proposition 1.8) and be able to give examples of subspaces U, V of $\mathbf{R}^3$ such that $\mathbf{R}^3 = U+V$, but the decomposition is not a direct sum (Use Proposition 1.9).

Chapter 2 **Fundamental concepts:** linear combination; span; finite-dimensional; linearly independent; basis; dimension.

Understand how to use the definition of linear independance to check that a list $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is linearly independent: write down the equation $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0$ (*) and manipulate it to yield $a_1 = 0, \ldots, a_n = 0$. In the special case where $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ are vectors in $\mathbf{F}^k$, i.e. k-tuples of elements of $\mathbf{F}$, then (*) gives a homogeneous system of k equations in the unknowns $a_1, \ldots, a_n$, and proving linear independence means proving that this system has only $(0, \ldots, 0)$ as solution.

Similarly use the definition of span to check that a list $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ of vectors in V spans V: take an arbitrary $\mathbf{v} \in V$, write the vector equation $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{v}$ (**) and manipulate it to exhibit field elements $a_1, \ldots a_n$ which work. In the special case where $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ are vectors in $\mathbf{F}^k$, i.e. k-tuples of elements of $\mathbf{F}$, then

$\mathbf{v} = (b_1, \ldots, b_k)$, and (**) gives a system of k equations in the unknowns $a_1, \ldots, a_n$; if this system has a solution, for every choice of $b_1, \ldots, b_k$, then the list spans.

The main content in this chapter is Theorem 2.6: *In a finite-dimensional V, the number of vectors in any linearly independent set is $\leq$ the number of vectors in any spanning set*, along with its Linear Dependence Lemma 2.4. Understand these proofs, and how the Theorem is used to show that any two bases for V have the same number of elements.

Theorems 2.10 and 2.12 are straightforward: their proofs essentially tell you how to implement their statements.

Understand the proof of Proposition 2.13 (*Every subspace U of a finite-dimensional V has a complementary subspace*, i.e $\exists W$ such that $V = U \oplus W$) well enough to be able to apply it as in Homework 1, Exercise 5iii. It involves finding a basis for U and then applying Theorem 2.12.

Chapter 3 **Fundamental concepts:** Linear map, null space, range, injective, surjective, matrix.

This chapter is less abstract that Chapter 2. Understand the concepts (in particular be able to prove that null T and range T are subspaces if T is linear). Understand how a linear map $T : V \to W$ is determined by its values on a list of basis elements. Understand the proof of Theorem 3.4, and understand what it means in terms of sets of linear equations with coefficients in a field $\mathbf{F}$ (pp. 47, 48). Understand that T has a matrix *once bases have been chosen for V and W* and that different bases will give different matrices for the same T.

Understand:
If $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$, $T(\mathbf{v}) = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \cdots + b_m\mathbf{w}_m$, and the matrix of T with respect to the bases $(\mathbf{v}_1, \ldots, \mathbf{v}_n), (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ is $[c_{ij}]$ (i.e. $T(\mathbf{v}_i) = c_{1i}\mathbf{w}_1 + c_{2i}\mathbf{w}_2 + \cdots c_{1m}\mathbf{w}_m$), then the column vector $\mathbf{b} = (b_1, \ldots, b_m)$ is the "matrix product" of $\mathbf{C} = [c_{ij}]$ with the the column vector $\mathbf{a} = (a_1, \ldots, a_n)$: $\mathbf{Ca} = \mathbf{b}$, i.e.

$$\left[\begin{array}{ccc} c_{11} & \cdots & c_{1n} \\ c_{21} & \dots & c_{2n} \\ \cdots & \cdots & \cdots \\ c_{m1} & \dots & c_{mn} \end{array}\right]\left[\begin{array}{c} a_1 \\ \cdots \\ \cdots \\ a_n \end{array}\right]=\left[\begin{array}{c} b_1 \\ \cdots \\ \cdots \\ b_m \end{array}\right].$$