Math 310: Homework 7

These exercises are adapted from Hoffman and Kunze, *Linear Algebra*

Ex 1. Recall that a function from $n \times n$ matrices to $\mathbb{F}$ is n-linear if it is linear in each row separately.

Each of the following expressions defines a function D on the set of 3×3 matrices over the set of real numbers. In which of these cases is D a 3-linear function?

(a) $D(A) = A_{11} + A_{22} + A_{33}$
(b) $D(A) = (A_{11})^2 + 3A_{11}A_{22}$
(c) $D(A) = A_{11}A_{12}A_{33}$
(d) $D(A) = A_{13}A_{22}A_{32} + 5A_{12}A_{22}A_{32}$
(e) $D(A) = 0$
(f) $D(A) = 1$.

Ex 2. Let $\mathbb{F}$ be a field. If A is a 2×2 matrix over $\mathbb{F}$, the **classical adjoint** of A is the 2×2 matrix $\text{adj}A$ defined by

$$\text{adj}A = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}.$$

Using $\det A = A_{11}A_{22} - A_{12}A_{21}$ as your definition of determinant, show that

(a) $(\text{adj}A)A = A(\text{adj}A) = (\det A)I$
(b) $\det \text{adj}A = \det A$
(c) $\text{adj}(A^t) = (\text{adj}A)^t$, where A^t, the **transpose** of A, is defined by $A^t_{ij} = A_{ji}$: It is A flipped across its main diagonal.

Ex 3. Let A be a 2×2 matrix over a field $\mathbb{F}$. Show that A is invertible if and only if $\det A \neq 0$. (Use Exercise 2). When A is invertible, give a formula for A^{-1}: write out the entries explicitly.

Ex 4. Define a function D on the 3×3 matrices with coefficients in $\mathbb{F}$ by

$$D(A) = A_{11} \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - A_{12} \det \begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} + A_{13} \begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}.$$

Show that D is alternating and 3-linear as a function of the columns of A.

Ex 5. Let D be an alternating n-linear function on the set of $n \times n$ matrices with coefficients in $\mathbb{F}$. Show that

(a) $D(A) = 0$ if one of the rows of A is 0.
(b) $D(B) = D(A)$ if B is obtained from A by adding a scalar multiple of one row of A to another.

Ex 6. The **characteristic polynomial** of a square matrix A is $f_A(\lambda) = \det(\lambda I - A)$, and its **characteristic equation** is $f_A(\lambda) = 0$. We have seen that the solutions of this equation are the eigenvalues of A. The object of this exercise is to prove, for

2×2 matrices, the *Cayley-Hamilton Theorem*: a matrix satisfies its own characteristic equation.

(a) Show that for a 2×2 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

the characteristic polynomial is $f_A(\lambda) = \lambda^2 - (A_{11} + A_{22})\lambda + \det A$.

(b) Show that $A^2 - (A_{11} + A_{22})A + (\det A)I = 0$ where this "0" is the 2×2 matrix with all entries 0.