Math 310 Spring 2008

Homework for week of 04/21

In text, do page 160 problems 21 and 22.

Ex 3 - corrected The *trace* of a matrix is the sum of all its diagonal elements: $\mathrm{trace}(A) = \sum_i a_{ii}$. Let V be the real vector space of all $n \times n$ matrices over $\mathbb{R}$, and given any two matrices $A, B \in V$ define

$$\langle A, B \rangle = \mathrm{trace}(AB) = \sum_{i,j} a_{ij} b_{ji}.$$

(i) Show that this satisfies all axioms for an inner product except possibly for positivity and nondegeneracy. (e.g. give an example (with $n = 2$) such that $A \neq 0$ but $\mathrm{trace} AA = 0$.)

(ii) A matrix A is *symmetric* if it is equal to its transpose , i.e. $a_{ji} = a_{ij}$. Show that if a real matrix A is symmetric, then $\langle A, A \rangle \geqq 0$, and $= 0$ if and only if A is the zero matrix. Thus $\mathrm{trace} AB$ defines an inner product on the space of real symmetric matrices.

(iii) Let V be the real vector space of $n \times n$ symmetric matrices. What is $\dim V$? What is the dimension of the subspace W consisting of those matrices A such that $\mathrm{trace}(A) = 0$? What is the dimension of the orthogonal complement $W^\perp$ relative to the inner product defined above?

Ex 4 Let A be an $n \times n$ matrix, and define $T \in \mathcal{L}(\mathbb{F}^n)$ by $Tv = Av$.

(i) Show that T is diagonalizable iff there exists an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.
(ii) How can you interpret the columns of the matrix Q? (Hint: think of these as vectors. What relation do they have to the operator T?)

Ex 5 Two linear operators S and T on a finite-dimensional vector space V are called *simultaneously diagonalizable* if there exists a basis $\mathcal{B}$ for V such that both $\mathcal{M}(S, \mathcal{B})$ and $\mathcal{M}(T, \mathcal{B})$ are diagonal matrices. This is equivalent to saying that there is a basis for V consisting of vectors that are eigenvectors for both S and T.

(i) Prove that if S and T are simultaneously diagonalizable operators then S and T commute. (Hint: see what the operators ST and TS do to a suitable basis for V.)

(ii) (Bonus) Prove also that if S and T are diagonalizable operators that commute then they are simultaneously diagonalizable.

(iii) Let $T_A, T_B \in \mathcal{L}(\mathbb{F}^n)$ be the operators defined by multiplication by the matrices A, B. Show that T_A, T_B are simultaneously diagonalizable iff there is an invertible matrix Q such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices. (cf Ex 4).

(iv) [has been deleted].