Stony Brook University
MAT 310 Linear Algebra **Fall 2008**
Review for Midterm 2

(References to Friedberg *et al.*, Linear Algebra, 4th Ed.)

2.5 Understand how change of bases affects the matrix $[T]_\beta^\gamma$ of a linear transformation $T\colon V \to W$. Namely if β' is a new basis for V and γ' a new basis for W write T as the composition $T = I_W \circ T \circ I_V$:

$$V \xrightarrow{I_V} V \xrightarrow{T} W \xrightarrow{I_W} W$$

going from V (basis β') to V (basis β) to W (basis γ) to W (basis γ'), so

$$[T]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'}[T]_\beta^\gamma[I_V]_{\beta'}^\beta.$$

Also, be able to calculate $[I_V]_{\beta'}^\beta$: suppose $\beta = (v_1, \ldots, v_n)$ and $\beta' = (v_1', \ldots, v_n')$; then the first column of $[I_V]_{\beta'}^\beta$ is the column of coefficients obtained when v_1' is written as a linear combination of $v_1, \ldots, v_n$, i.e. it's the vector v_1' written in the basis $(v_1, \ldots, v_n)$. Similarly second column of $[I_V]_{\beta'}^\beta$ is the vector v_2' written in the basis $(v_1, \ldots, v_n)$, etc. This is especially simple when β is the standard basis.

On the other hand if you know $[I_W]_{\gamma'}^\gamma$, the matrix $[I_W]_\gamma^{\gamma'}$ can be retrieved by inverting $[I_W]_{\gamma'}^\gamma$, since $[I_W]_\gamma^{\gamma'}[I_W]_{\gamma'}^\gamma = [I_W]_\gamma^\gamma = I$, the identity matrix. *Examples 1, 2; Corollary, p.115; Problem 6d*

3.1 Understand the three kinds of elementary row operations, and how each of them can be carried out on a matrix A by *left*-multiplying A with the appropriate *elementary matrix* (which is the matrix obtained by applying that row operation to the identity matrix!). *Theorem 3.1, Example 2*. Be able to invert an elementary matrix on inspection (be able to prove *Theorem 3.2*; pay attention to type 3.) Understand this paragraph when "row" → "column" and "left" → "right." *Problems 2, 4, 7*

3.2 Remember that the *rank* of a linear transformation is the dimension of its range. Understand that the rank of an $m \times n$ matrix A is the rank of $L_A\colon \mathbf{F}^n \to \mathbf{F}^m$, and *Theorem 3.3*. Understand why the rank of A is the maximum number of linearly independent columns in A (*Theorem 3.5, Examples 1, 2*). Understand the content of *Theorem 3.6*: After an appropriate change of basis, an arbitrary linear $T\colon \mathbf{F}^n \to \mathbf{F}^m$ becomes $(a_1, a_2, \ldots, a_r, a_{r+1}, \ldots, a_n) \to (a_1, a_2, \ldots, a_r, 0, \ldots, 0)$ where r is the rank of T. Understand how this theorem and its proof imply *Corollary 2: rank*(A^t) = *rank*(A) and *Corollary 3: Every invertible matrix is a product of elementary matrices* which are both very useful.

Know how to compute the inverse of an invertible matrix A by forming the augmented matrix A, I and row-reducing it to get I, A^{-1} *Examples 5, 6*: if A cannot be row-reduced to I, then A is not invertible.

4.1 Understand how to compute the determinant of a 2×2 matrix $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc$, and the connection between determinants and area: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$ the area of the parallelogram spanned by the vectors (a, b) and (c, d) with a plus sign if $((a, b), (c, d))$ is a right-handed system and minus otherwise. *Problems 2, 3, 4.*

4.2 Understand how to carry out the inductive calculation of the determinant of a large square matrix, and be able to do it for 3×3 and 4×4 matrices. *Examples 1, 2, 3.* Understand the proof of *Theorem 4.3: the function* $A \to \det A$ *is linear in each row separately.* Understand the statement of the *Lemma* on pp. 213-214, and how it and Theorem 4.3 imply *Theorem 4.4: determinant can be calculated by cofactor expansion along any row.* Understand how this implies that if two rows of A are identical, then $\det A = 0$; and furthermore that if A' is derived from A by interchanging 2 rows, then $\det A' = -\det A$. Be able to use these concepts to prove *Theorem 4.6:* det *is invariant under type-3 elementary row operations* and its *Corollary p. 217: if A is $n \times n$ and rank*$(A) < n$ *then* $\det A = 0$. Finally be able to calculate the determinant of a large matrix by using row operations to simplify the problem *Examples 5,6. Problems 9,21.*

4.3 Know how to prove the important *Theorem 4.7:* $\det(AB) = \det A \cdot \det B$ and the related *Theorem 4.8:* $\det A^t = \det A$ (A^t the transpose). *Problems 8,9.*