MAT 310 Fall 2008 Review for Midterm 1

1.2 Fundamental concepts. $\mathbf{F}$ is a field (check Appendix C if you are not completely comfortable with what a field is; main examples $\mathbf{Q}, \mathbf{R}, \mathbf{C}$, but also $\mathbf{Z}_2$ and $\mathbf{Z}_p$, p prime; understand why $\mathbf{Z}$ is not a field and why $\mathbf{Z}_4$ is not a field.); V is a vector space over $\mathbf{F}$. *Example 2, Example 4.* Check *Examples 6, 7* for things that are *not* vector spaces. Understand how the "Cancellation Law" follows from the axioms (Theorem 1.1) and how to derive the basic facts in Theorem 1.2. Review homework exercises and 16, 17.

1.3 Subspaces. Understand *Examples 3, 4.* Be able to prove Theorem 1.4. Understand *Exercise 8,abde.* Be able to prove the statement in *Exercise 20.*

1.4 Linear combinations. The fundamental concept in Linear Algebra! Be able to work an example like *Example 2*: checking whether a given vector v is a linear combination of some other vectors $w_1, w_2, \ldots, w_k$ always reduces to seeing if a certain set of linear equations has a solution. *Exercise 3.*

1.5 Linear dependence and independence. Be able to work an example like *Example 1*: checking whether a set of vectors $v_1, v_2, \ldots, v_k$ is linearly independent always reduces to seeing if a certain set of equations has a *non-zero* solution. *Exercise 3.* Be able to prove Theorem 1.6 (a subset of a linearly independent set is linearly independent). Understand the proof of Theorem 1.7, paying attention to the details. *Exercise 15.*

1.6 Bases and dimension. Understand the definition "β is a basis of the vector space V." And how it is equivalent to the statement in Theorem 1.8. Be able to prove Theorem 1.9: If a vector space V is generated by some finite set S, then some subset of S is a basis for V. ("generates" defined on page 30). Checking that a set of vectors is a basis for V always means checking two things: the set is linearly independent and it spans V. *Exercises 8,9.*

2.1 Linear transformation, null space, range. Know the definition of linear transformation, and understand why T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for every $x, y \in V$ and every $c \in \mathbf{F}$ *Exercise 7.* Know some examples, like *Examples 2-7.* Know for $T\colon V \to W$ the definition of the null-space (also, "kernel") $\mathrm{N}(T)$ and the range (also, "image") $\mathrm{R}(T)$ and be able to prove that if $T\colon V \to W$ is linear, then $\mathrm{N}(T)$ is a subspace of V and $\mathrm{R}(T)$ is a subspace of W. Understand the Dimension Theorem (Theorem 2.3) and be able to apply it to prove Theorem 2.5. Be able to prove Theorem 2.6 (a linear transformation is completely determined by its values on a set of basis vectors, and those values can be arbitrarily chosen). *Exercises 10, 11, 12.*

2.2 Matrix representation of a linear transformation. Understand the concept of *ordered basis*, and why the order of the basis vectors matters in setting up the matrix representation. (What happens to the matrix $[T]_\alpha^\beta$ if the order of the elements of α is changed? if the order of the elements of β is changed?). Know what is meant by the "standard ordered basis" for $\mathbf{F}^n$. Be able to write down the entries of $[T]_\alpha^\beta$ given $V, W, T\colon V \to W, \alpha$ and β. Understand (Theorem 2.7) why the set $\mathcal{L}(V, W)$ of linear transformations from V to W is itself a vector space, with the correct definitions of sum and scalar product. Be able to prove Theorem 2.8 and be able to explain, using the language of section 2.4, that once ordered bases $\alpha = (v_1, \ldots, v_n)$ for V and $\beta = (w_1, \ldots, w_m)$ for W have been chosen, the map $[]_\alpha^\beta\colon \mathcal{L}(V, W) \to M_{m\times n}(\mathbf{F})$, taking $T\colon V \to W$ to $[T]_\alpha^\beta$, is a vector-space isomorphism.

2.3 Composition of linear transformations, matrix multiplication. Understand that when $T\colon V \to W$ and $U\colon W \to Z$ are composed as functions (i.e. $UT(v) = U(T(v))$), if T and U are linear so is their composition UT. Know the definition of matrix multiplication (page 87), and be able to show that if α and β as above are orered bases for V and W, and if $\gamma = (x_1, \ldots, x_p)$ is an ordered basis for Z then $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$ (Theorem 2.11).

Understand also that the i-th column of $[T]_\alpha^\beta$ is made up of the coefficients $a_{ji}, j = 1 \ldots m$ occurring when $T(v_i)$ is written as a linear combination $a_{1i}w_1 + \cdots + a_{mi}w_m$, and that if $v = c_1v_1 + \cdots c_nv_n$ is an arbitrary vector in V, then the coefficients of $T(v) = d_1w_1 + \cdots d_mw_m \in W$ are given by the matrix product $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$, where $[v]_\alpha$ is the *column vector* ($n \times 1$ matrix) $(c_1, \ldots, c_n)$ and $[T(v)]_\beta$ is the column vector ($m \times 1$ matrix) $(d_1, \ldots, d_m)$. *Exercises 12, 14ab.*