MAT 200: HANDOUT 1
LOGICAL AXIOMS AND INFERENCE RULES

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1. Propositional logic

Axioms: all tautologies

Rules of inference:

- Propositional consequence (see definition on page 38)
- **Conditional rule:**

$$\frac{\begin{array}{l}\text{Assume } P\\ \text{[some intermediate steps]}\\ Q\end{array}}{P \to Q}$$

  Here, intermediate steps can use P, so they are only true under the additional assumption P; however, the conclusion $P \to Q$ is valid without any assumptions.

Some frequently used tautologies.

$$\begin{array}{ll}\sim(P \wedge Q) \leftrightarrow (\sim P \vee \sim Q) & \\ \sim(P \vee Q) \leftrightarrow (\sim P \wedge \sim Q) & \text{(De Morgan's laws)}\\ (P \to Q) \leftrightarrow (\sim Q \to \sim P) & \\ \sim(P \to Q) \leftrightarrow (P \wedge \sim Q) & \end{array}$$

Some frequently used rules. This is not a complete list — there are infinitely many rules that can be obtained from general rule of propositional consequence — but just the most frequently used ones. See also Note 1.

Name	Formula	Verbal argument
Modus Ponens	P $\underline{P \to Q}$ Q	If P is true, then Q is true. P is true. Therefore, Q is true
Modus Tollens	$P \to Q$ $\underline{\sim Q}$ $\sim P$	If P is true, then Q is true. But Q is false. Thus, P must be false.
Transitivity of implication	$P \to Q$ $\underline{Q \to R}$ $P \to R$	
Conjunction	P $\underline{Q}$ $P \wedge Q$	
Biconditional	$P \to Q$ $\underline{Q \to P}$ $P \leftrightarrow Q$	Assume that P is true. Then Q is true. Conversely, assume that Q is true. Then P is true. Therefore, Q is true if and only if P is true.
Proof by cases	$P_1 \vee P_2$ $P_1 \to Q$ $\underline{P_2 \to Q}$ Q	Consider the following two cases: • P_1 is true. Since $P_1 \to Q$, this implies Q. • P_2 is true. Since $P_2 \to Q$, this implies Q. Thus, in both cases Q is true. Since at least one of P_1, P_2 is true, this means that Q must be true.
Proof by contradiction (indirect proof)	$\underline{\sim Q \to \text{contradiction}}$ Q (Contradiction is a formula which is always false, i.e. a negation of a tautology)	Assume that Q is false. Then, after some steps, we get a cotnradiction. Thus, our assumption is wrong, i.e., Q is true.
Substitution	$P \leftrightarrow Q$ $\underline{S[P]}$ $S[P/Q]$ where $S[P]$ is a formula containing one or more occurence of P as subformulas, and $S[P/Q]$ is result of replacing in S some of occurences of P by Q (see Note 3).	

2. Axioms and rules involving quantifiers

Axioms.

De Morgan laws:

$$(\sim\forall x\ P) \leftrightarrow (\exists x\ \sim P)$$
$$(\sim\exists x\ P) \leftrightarrow (\forall x\ \sim P)$$

Universal Specification (US):

$$\forall x\ P(x) \to P(t)$$

where t is any term (i.e., something constructed out of variables, constants, and operations) of the same type as x. (See also Note 3.)

Existential Generalization (EG):

$$P(t) \to \exists x\ P(x)$$

where t is as in US axiom.

Inference rules.

Name	Formula	Verbal argument
Universal Generalization (UG)	[Proof with no assumptions on x] $P(x)$ $\forall x\ P(x)$	Take an arbitrary value of x. Then we get $P(x)$. Since we used no assumptions about the value of x, this can be repeated for any value of x. Thus, $\forall x\ P(x)$
Existential Specification (ES)	$\exists x\ P(x)$ $P(c)$ where c is a new constant symbol, which had not been used before (see also Note 4).	There is a value of x for which $P(x)$ is true. Let c be such a value, so that we have $P(c)$

3. Notes

(1) This axiomatization of laws of logic is not the only one possible one: there are many other ways to give a list of axiosm and rules, giving equivalent results. In fact, one can get a much shorter list of axioms: for example, the Conditional Rule can be deduced from propositional consequence, so it is not necessary to include it. However, allowing to use it directly makes proofs much shorter.

(2) It is common to combine several of these rules in one step. For example, we can combine Proof by Contradiction with Conditional Rule to get the following variant of Proof by Contradiction:

Assume $\sim Q$
[some intermediate steps]
contradiction
Q

(3) Strictly speaking, Universal Specification axiom only works under an additional assumption that substituting t for x in $P(x)$, we do not get a conflict of variables: otherwise, we could use it to deduce from (correct) formula $\forall x\ \exists y\ y > x$, by letting $x = y$, an incorrect formula $\exists y\ y > y$. Precise definition of "conflict of variables" can be found in the book. Similarly, Substitution Rule only works if replacing P by Q does not lead to a conflict of variables.

(4) When using Existential Specification rule, it is important to note that c must be a constant (i.e., you can not change its value later — in particular, you can not apply UG rule to it) and that you must use for it a symbol **which had not been used before**. In particular, if you need to use ES rule twice, you need to choose two different symbols: you can't use c in both of them – take, e.g., c and d, or c_1 and c_2.