MAT 200
SOLUTIONS TO HOMEWORK 9

NOVEMBER 23, 2004

Section 5.2: 3, 5, 13

3 (a) Suppose $A \cap A'$ is not empty. Then there exists a $x \in A \cap A'$. So x must be $x \in A$ and $x \in A'$. But $x \in A'$ means $x \notin A$, and this is a contradiction since we have $x \in A$ and $x \notin A$.

(b) Since the universal set contains A, A', it also contains their union: $A \cup A' \subseteq U$. Now for the reverse inclusion, let $x \in U$. Since any element is either in or not in A, $x \in A$ or $x \in A'$. So $x \in A \cup A'$. So we have established $x \in U \leftrightarrow x \in A \cup A'$. By UG we have $U = A \cup A'$.

5 (a) Suppose $x \in A$. Then $x \in A$ or $x \in B$ (tautology). So $x \in A \cup B$.

(c) Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$. So $x \in A$. By UG we have $A \cap B \subseteq A$.

13 When doing proof by cases, we need to check that the cases we consider cover all possibilities, i.e., at least one of them always holds. In this proof, it is false: neither case covers possibility $x = b$. In fact, the statement itself is false: $(a, b) \cup (b, c) \neq (a, c)$ because $b \in (a, c)$ but $b \notin (a, b) \cup (b, c)$.

Section 5.3: 1 a,e, 7, 11

1 a,e (a) $\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

(e) Since $\mathcal{P}(\{1, 3\}) = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$, subtracting this from a) we get $\{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$.

7 By definition,

$$
\begin{aligned}
x \in B \cap \left(\bigcup A_i\right) &\leftrightarrow (x \in B) \wedge (x \in \bigcup A_i) \\
&\leftrightarrow (x \in B) \wedge (\exists i \; x \in A_i) \\
&\leftrightarrow \exists i \; (x \in B) \wedge (x \in A_i) \qquad \text{By Law of logic (18) on p. 102} \\
&\leftrightarrow \exists i \; x \in (B \cap A_i) \leftrightarrow (x \in \cup(B \cap A_i))
\end{aligned}
$$

Alternative proof:

Forward direction: assume $x \in B \cap \left(\bigcup A_i\right)$. Then $x \in B$ and $\exists i \; x \in A_i$. Let i_0 be such a value of i, so that $x \in A_{i_0}$. Then $x \in (B \cap A_{i_0})$, so $\exists i \; x \in (B \cap A_i)$, which is equivalent to $x \in \bigcup(B \cap A_i)$.

Reverse direction: assume $x \in \bigcup(B \cap A_i)$. Then $\exists i \; x \in (B \cap A_i)$. Let i_0 be such a vlaue of i, so that $x \in (B \cap A_{i_0})$. Then $x \in B$ and $x \in A_{i_0}$. Thus, $x \in \bigcup A_i$. Since we also have $x \in B$, it means that $x \in B \cap \left(\bigcup A_i\right)$.

11 $A_1 = [2^{-1}, 2^0) = [\frac{1}{2}, 1)$. $A_1 \cup A_2 = [\frac{1}{4}, 1)$, $A_1 \cup A_2 \cup A_3 = [\frac{1}{8}, 1)$.

Repeating this, we can prove that

$$A_1 \cup \cdots \cup A_n = [\frac{1}{2^n}, 1)$$

Now we claim that

$$\bigcup_{n=1}^{n=\infty} A_n = (0,1)$$

Indeed: if $x \in \bigcup_{n=1}^{n=\infty} A_n$, then $x \in [\frac{1}{2^n}, \frac{1}{2^{n-1}})$ for some n, so $0 < x < 1$. Thus, $\bigcup_{n=1}^{n=\infty} A_n \subseteq (0,1)$. In particular, 0 is excluded because $0 \notin A_n$ for any n.

Conversely, let $0 < x < 1$. Since $\lim \frac{1}{2^n} = 0$, it means that there exists n such that $\frac{1}{2^n} < x < 1$. Thus, $\exists n\ x \in A_1 \cup \cdots \cup A_n$. Thus, $x \in \bigcup_{n=1}^{n=\infty} A_n$. This shows that $(0,1) \subseteq \bigcup_{n=1}^{n=\infty} A_n$.

Combining these two steps, we see that

$$\bigcup_{n=1}^{n=\infty} A_n = (0,1)$$

So

$$[0,1] - \bigcup_{n=1}^{n=\infty} A_n = \{0,1\}$$

NOTE: just writing $A_1 \cup \cdots \cup A_n = [\frac{1}{2^n}, 1)$ and saying "let us take limit as $n \to \infty$" is not a legal proof. To do it, you firts need to explain what a limit of a sequence of sets is — which we never did, and have no intention of doing. The definition of infinite union did not involve any limits.

Section 6.1: 4 abe, 5

4 If we denote by $|M|$ the number of elements in set M, then by Theorem 5.8, the number of elements in $\mathcal{P}(M)$ is $2^{|M|}$.

(a) $|A \times B| = 2 \times 3 = 6$, so $|\mathcal{P}(A \times B)| = 2^6$.

(b) $|\mathcal{P}(A) \times \mathcal{P}(B)| = |\mathcal{P}(A)| \times |\mathcal{P}(B)| = 2^3 \times 2^2 = 2^5$

(e) $|B \times B \times B \times B| = |B| \times |B| \times |B| \times |B| = 2^4$

5 (a) $(D - \{0\}) \times L \times L \times D \times D$

(b) $9 \times 26 \times 26 \times 10 \times 10 = 608400$