MAT 200
COURSE NOTES ON GEOMETRY

STONY BROOK MATHEMATICS DEPARTMENT

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1. Introduction

1.1. References. The exposition of Euclidean geometry given here is a significantly abridged and slightly modified version of the one given in the book *The Foundations of Geometry and the Non-Euclidean Plane*, by George E. Martin. This book is on reserve in Math-Phys library. The exposition in this book in turn follows the one suggested in 1932 by G. D. Birkhoff. However, this approach —as well as virtually all other known approaches to axiomatization of plane geometry—is based on the famous Euclid's *Elements*, written about 300 BC and used as the primary textbook in geometry for the next 2000 years.

1.2. Euclidean geometry as axiomatic theory. The theory tries to describe geometric properties of various figures in the plane. Figures are understood as sets of points; we will use capital letters for points and write $P \in m$ for "point P lies in figure m", or "figure m contains point P". The notion of "point" can not be defined: it is so basic that it is impossible to explain it in terms of simpler notions. In addition, there are some other basic notions (lines, distances, angles) that can not be defined. Instead, we can postulate some properties of these objects; these "postulates" are usualy called "axioms of Euclidean geometry". **All results in Euclidean geometry should be proved by deducing them from the axioms**; justifications "it is obvious", "it is well-known", or "it is clear from the figure" are not acceptable. We allow use of all logical rules and axioms. We also assume knowledge of real numbers and their properties.

There are many ways to give a set of axioms for Euclidean geometry; different books use slighlty different sets of axioms. So be careful when comparing these notes to any other books.

1.3. Basic objects. These objects are the basis of all our constructions: all objects we will be discussing will be defined in terms of these objects. No definition is given for these basic objects.

- Points
- Lines
- Distances: for any two points A, B, there is a non-negative real number $|AB|$, called distance between A, B
- Angle measures: for any angle $\angle ABC$, there is a real number $m\angle ABC$, called the measure of this angle (more on this later).

2. Incidence Axioms

In this section, we introduce the first axioms which deal with lines, points, and relation "point P lines on line l" (this is sometimes called *incidence* relation, hence the name of this section). We do not discuss distances or angles here; they will be discussed in later axioms.

2.1. Notation.

Definition 2.1. Two lines l, m are called intersecting if they are distinct ($l \neq m$) and have at least one common point.

Two lines l, m are called parallel if $l = m$ or l, m have no common points. Notation: $l \| m$.

Exercise 2.1: Show that two lines are parallel iff they are not intersecting, and that any two lines are either parallel or intersecting.

Exercise 2.2: Show that $l \| m \leftrightarrow m \| l$.

2.2. Incidence and Parallel lines axioms.

Incidence Axiom. (1) *For any two distinct points, there is a unique line that contains these two points*

(2) *For any line, there exists a point not on this line.*

We will denote the unique line containing points A, B by $\overleftrightarrow{AB}$.

Parallel Lines Axiom. *For any line l and a point P not on l, there exists a unique line containing P and parallel to l.*

2.3. First theorems.

Theorem 2.1. *If two distinct lines intersect, then they intersect at exactly one point.*

Theorem 2.2. *For any lines l, m, n, if $l \| m$ and $m \| n$, then $l \| n$.*

Exercise 2.3: Let A, B, C be distinct points such that C lies on the line $\overleftrightarrow{AB}$. Show that then A lies on the line $\overleftrightarrow{BC}$.

Exercise 2.4: Lel l, m, n be lines such that $l \| m$ and n intersects l. Show that then n must also intersect m.

2.4. Historical notes. Historically, the Parallel Lines axiom (or, in original Euclid's numbering, fifth postulate) was the most controversial. It is the least obvious one, so there have been many attempts to deduce it from the remaining axioms. Only in the end of 19th century it was shown (by Bolyai and Lobachevskii) that this postulate can not be deduced from the remaining axioms. They did it by constructing a different kind of "plane", with different notions of lines, distances, and angles, in which all other axioms hold but the axiom of parallel lines fails. Nowadays such a plane is usually called "non-Euclidean plane", or "Lobachevskii plane".

3. Ruler Axiom

In this section we give the axiom which describes properties of distance and order relation for points on a line.

3.1. Ruler axiom.

Ruler Axiom. *Let l be a line and P a point on l. Then there is an identification $f\colon l \simeq \mathbb{R}$ such that $f(P) = 0$ and for any points A, B on l, $|AB| = |f(A) - f(B)|$.*

This axiom shows that any line "looks like" the usual number line, $\mathbb{R}$. This allows us to use many known properties of $\mathbb{R}$ to prove various results about points on lines.

An identification $f\colon l \simeq \mathbb{R}$ as in the Ruler Axiom is called a coordinate system on l. Note that it is not unique: there are many coordinate systems on a given line.

Theorem 3.1. *Let A, B be distinct points. Then there exists a coordinate system f on the line $\overleftrightarrow{AB}$ such that $f(A) = 0$, $f(B) > 0$.*

3.2. Order on a line.

Definition 3.1. Let A, B, C be points on a line l. We say that B lies between A and C (notation: $A - B - C$) if there is a coordinate system on l such that $f(A) < f(B) < f(C)$ or $f(C) < f(B) < f(A)$.

We can use familiar properties of order on $\mathbb{R}$ to prove various results about relative position of points on a line. Most of them are very easy to prove.

Theorem 3.2. (1) *For any three distinct points on a line l, exactly one of them lies between two others.*
(2) *If $A - B - C$ and $B - D - C$, then $A - D - C$.*

Definition 3.2. Let A, B be distinct points. Then the segment AB is the set of all points C on the line $\overleftrightarrow{AB}$ such that C is between A and B.

Note that according to this definition, the endpoints A, B are not included in AB.

Theorem 3.3. *Let V be a point on the line l. Then V divides l into two pieces, which are called rays, or half-lines.*

To make this precise, we need to explain the meaning of "divides into two pieces". This can be explained as follows: if $A, B \in l$ are in different half-lines, then V is between A and B; if A, B are in the same half-line, then V is not between A, B. (make a picture to convince yourself).

Proof. Choose a coordinate system on l such that $f(V) = 0$ (by Ruler Axiom, this is possible). Define the first half-line to consist of "positive" points (i.e., those with $f(A) > 0$, and the other of "negative" ones (i.e., with $f(A) < 0$). □

Notation: let V, A be distinct points. Then V divides line $\overleftrightarrow{VA}$ into two rays. We will denote by $\overrightarrow{VA}$ the ray that contains point A.

Theorem 3.4. *Let $\overrightarrow{VA}$ be a ray, and $B \in \overrightarrow{VA}$. Then $\overrightarrow{VB} = \overrightarrow{VA}$.*

3.3. PROPERTIES OF DISTANCE. All properties discussed here are corollaries of the Ruler Axiom.

Theorem 3.5. *For any A, B, $|AB| \geq 0$. Also, $|AB| = 0$ iff $A = B$.*

Theorem 3.6. *Let A, B, C be distinct points such that B is between A and C. Then $|AB| + |BC| = |AC|$.*

Exercise 3.1: Let $\overrightarrow{VA}$ be a ray, and r —a positive real number. Show that there is a unique point P on the ray $\overrightarrow{VA}$ such that $|VP| = r$.

Exercise 3.2: If B is a point in the ray $\overrightarrow{VA}$, and $|VB| < |VA|$, then B lies between V and A.

Exercise 3.3: Let A, B be distinct points. Prove that there exists a unique point M on the segment AB such that $|AM| = |MB|$ (this point is called the midpoint of the segment AB).

4. Protractor Axiom

In this section, we give axioms relevant to measurement of angles. However, before we do it we need to introduce a technical axiom which introduces the notion of a halfplane.

4.1. Plane separation axiom.

Plane Separation Axiom. *Let l be a line. Then it divides the plane into two halfplanes so that*

- *If A, B are in different halfplanes, then segment AB intersects line l.*
- *If A, B are in the same halfplane, then segment AB does not intersect line l.*

If A, B are in the same halplane, we will also say that they are on the same side of l.

4.2. Angles and angle measures.

Definition 4.1. An angle is the figure consisting of a point A and two distinct rays starting at A. Notation: an angle formed by rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is denoted by $\angle BAC$. We may also use notation $\angle A$ if it is clear from the context what rays form the sides of the angle.

One of the basic notions we have is the notion of angle measure: for each angle $\angle ABC$, we have a positive real number $m\angle ABC$ called measure of $\angle ABC$. As before, we can not give a definition of this measure. Instead, the Protractor Axiom below gives some properties of the angle measure. It is common to use Greek letters $\alpha, \beta, \gamma, \ldots, \varphi, \theta$ for angle measures.

4.3. Historical note. The use of the phrase "measure of an angle" is relatively modern. Up to about 50 years ago, the measure of the angle at A was simply denoted by A or $\angle A$, and it was left to the reader to distinguish between the angle and its measure. When convenient, we will follow this convention, and use the same notation for an angle and its measure.

4.4. Protractor axiom.

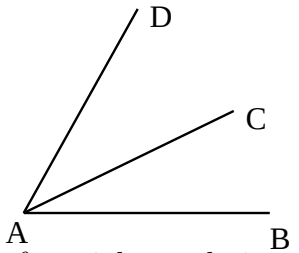
Protractor Axiom. (1) *For any angle $\angle BAC$, $0 < m\angle BAC \leq \pi$.*

(2) *If $\angle BAC$ is a straight angle (i.e., A, B, C lie on a line with A between B and C), then $m\angle BAC = \pi$.*

(3) *Let A, B be distinct points, and H — one of halfplanes into which $\overleftrightarrow{AB}$ divides the plane. Then for any $\alpha \in \mathbb{R}, 0 < \alpha < \pi$ there exists a unique ray $\overrightarrow{AC}$ in the halfplane H such that $m\angle BAC = \alpha$.*

(4) *If ray $\overrightarrow{AC}$ lies inside angle $\angle BAD$, then $m\angle BAD = m\angle BAC + m\angle CAD$ (see figure).*

Part (4) talks about a ray being "inside" an angle. This notion can be accurately defined using the Plane Separation Axiom, but it is cumbersome. Therefore, as a compromise between rigor and readability we will just refer to the figure at right.



Note that we measure the angles in radians, so that the measure of straight angle is π rather than 180. Also, we always measure the smaller of the two sectors formed by two rays, so the measure of any angle is at most π.

Exercise 4.1: Let A, B be distinct points, and H — one of the halfplanes into which $\overleftrightarrow{AB}$ divides the plane. Show that for any real numbers r, α, with $r > 0, 0 < \alpha < \pi$, there exists a unique point C in H such that $|AC| = r$ and $m\angle BAC = \alpha$. Please note that you can only use the results we have proved: in particular, we do not yet know anything about circles.

4.5. Interior of an angle. We will frequently use the notion of a point or a ray "lying inside" an angle. This can be precisely defined using the notion of halfplane, and then we could use Plane Separation Axiom and Protractor Axiom to prove various results about this notion, much as we have used Ruler Axiom to prove various results about the relation "to lie between" for points on a line. However, it would take too much time, and since most of the results are rather geometrically obvious, we will just state them. Proofs of the results below can be found, e.g., in the book by Martin referenced in the introduction.

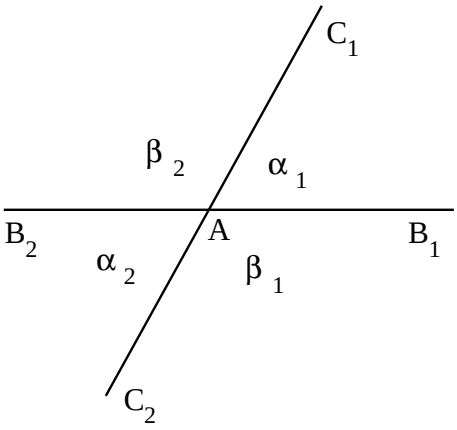
Theorem 4.1 (Crossbar theorem)**.** *The ray $\overrightarrow{AD}$ is inside the angle $\angle BAC$ if and only if $\overrightarrow{AD}$ crosses the segment BC.*

Theorem 4.2 (Monotonicity of angles)**.** *Let A, B, C, D be distinct points such that C and D lie on the same side of the line $\overleftrightarrow{AB}$. Then $m\angle BAD < m\angle BAC$ iff $\overrightarrow{AD}$ is inside the angle $\angle BAC$.*

Exercise 4.2: Show that without the assumption that C, D lie on the same side of $\overleftrightarrow{AB}$, Theorem 4.2 would be false.

Exercise 4.3: Show that for any angle $\angle BAC$, there is a unique ray $\overrightarrow{AD}$ inside it so that $m\angle BAD = m\angle DAC$. This ray is called the *bisector* of angle $\angle BAC$.

4.6. Vertical and supplementary angles. Let l, m be distinct lines intersecting at point A. Then these lines define four angles as shown in the figure below (again, this can be proved but we omit the proof). In this situation, two angles are called supplementary if they have a common side; otherwise, they are called vertical. Thus, in the figure below angles $\angle B_1AC_1$ and $\angle C_1AB_2$ are supplementary, while $\angle B_1AC_1$ and $\angle B_2AC_2$ are vertical.



Theorem 4.3. (1) *Sum of measures of supplementary angles is equal to π.*
(2) *Measures of vertical angles are equal.*

Proof. (1) By part (4) of the Protractor Axiom, sum of measures of supplementary angles is equal to the measure of the straight angle. By part (b) of the same axiom, the measure of the straight angle is π.
(2) Let α_1, α_2 and β_1, β_2 be two pairs of vertical angles as in the figure above. Then by part (a), $\alpha_1 + \beta_1 = \pi$. But also by part (a), $\alpha_2 + \beta_1 = \pi$. Subtracting these equalities, we get $\alpha_1 = \alpha_2$. In a similar way one proves that $\beta_1 = \beta_2$.

□

This result shows that when we have two intersecting lines, they define two different angles, α and β such that $\alpha + \beta = \pi$. Thus, when talking about "angle between lines", we need to specify which of these angles is being used.

5. Triangles

5.1. Basics. A triangle is a figure consisting of three points, A, B, C, not lying on one line, and the three segments connecting them, AB, BC, AC. Points A, B, C are called vertices of the triangle; segments AB, BC, AC, sides. A triangle with vertices A, B, C is denoted $\triangle ABC$.

Each triangle defines three angles, $\angle BAC, \angle ABC, \angle BCA$. It is also common to use shorter notation $\angle A, \angle B, \angle C$ if it is clear from the context what triangle is being discussed.

Thus, every gives six real numbers: measures of the three angles and lengths of the three sides. It is common to denote $\alpha = m\angle A, \beta = m\angle B, \gamma = m\angle C$ and $a = |BC|, b = |AC|, c = |AB|$

This definition is a way to formalize our intuitive picture of a triangle as something built out of three sticks joined together at the ends.

5.2. Congruence.

Definition 5.1. Two triangles, $\triangle ABC$ and $\triangle A'B'C'$, are congruent if the corresponding angles have equal measures, and the corresponding sides have equal lengths. That is, the triangles, $\triangle ABC$ and $\triangle A'B'C'$ are congruent if $m\angle A = m\angle A'$; $m\angle B = m\angle B'$; $m\angle C = m\angle C'$; $|AB| = |A'B'|$; $|AC| = |A'C'|$; and $|BC| = |B'C'|$. In this case, we write $\triangle ABC \cong \triangle A'B'C'$.

Please note that notation $\triangle ABC \cong \triangle A'B'C'$ not only says that the two triangles are congruent but also says that they are congruent in such a way that vertex A corresponds to vertex A', B to B', and C to C'.

If you imagine a triangle as a physical object, constructed of sticks joined at ends, then two triangles are congruent if you can put one on top of another so that they exactly match. Note that you are allowed to turn a triangle "face down" in the process.

5.3. The Angle-Side-Angle congruence axiom.

Angle-Side-Angle Congruence Axiom. *If $m\angle A = m\angle A'$, $m\angle B = m\angle B'$ and $|AB| = |A'B'|$, then $\triangle ABC \cong \triangle A'B'C'$.*

This axiom is commonly referred to as ASA axiom.

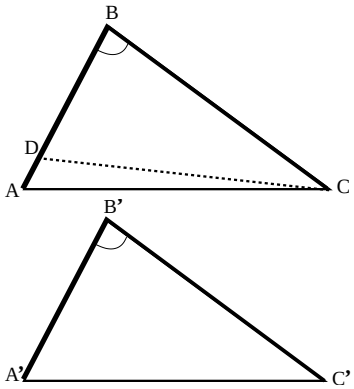
One can also try other ways to define a triangle by three pieces of information, such as three sides (SSS), three angles (AAA), or two sides and an angle. For the two sides and an angle, there are two versions, one in which the two sides are adjacent to the given angle (SAS) and the other in which one of the given sides is opposite to the given angle (SSA).

Exercise 5.1: Convince yourself that SSS and SAS do define a triangle up to congruence, but AAA and SSA do not. (We currently do not have enough tools to prove this rigorously, so at best you can give a convincing figure.)

5.4. Congruence via SAS.

Theorem 5.1 (SAS)**.** *If $\triangle ABC$ and $\triangle A'B'C'$ are such that $|AB| = |A'B'|$, $m\angle ABC = m\angle A'B'C'$, and $|BC| = |B'C'|$, then they are congruent.*

Proof. Suppose we are given two triangles $\triangle ABC$ and $\triangle A'B'C'$ as in the statement. If $m\angle BCA = m\angle B'C'A'$, then we would be done (by ASA).



So let us consider the case where they are different, and arrive at a contradiction. We may assume that $m\angle BCA > m\angle B'C'A'$ (if not, just exchange the names on the triangles).

Apply the Protractor Axiom to find a ray $\overrightarrow{AD}$ where some point D lying between A and B, so that $m\angle BCD = m\angle B'C'A'$.
By ASA, $\triangle BCD \cong \triangle B'C'A'$. Therefore $|DB| = |A'B'|$. But we are given that $|A'B'| = |AB|$. Therefore, $|DB| = |AB|$. Since D lies on the line determined by A and B, and lies between them, this contradicts Ruler Axiom.

□

Exercise 5.2: The proof above has minor gaps: when introducing ray $\overrightarrow{AD}$, it uses not only Protractor Axiom but also (implicitly) various properties of angles and the relation "to lie between" for points on a line. Write explicitly all the results it uses, and complete the proof by showing how these results follow from various axioms and theorems given above.

Exercise 5.3: Let A, B, C, D be points such that no three of them lie on a line, the segments AC and BD intersect, and the intersection point M is the midpoint (see Exercise 3.3) for each of them. (Later we will show that this is equivalent to saying that $ABCD$ is a parallelogram.) Show that

(1) $\triangle AMD \cong \triangle CMB$
(2) $|AD| = |BC|, |AB| = |CD|$
(3) $m\angle ABD = m\angle BDC$
(4) $m\angle ABC = m\angle ADC$.

5.5. Isosceles triangles. A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

Theorem 5.2 (Base angles equal)**.** *If $\triangle ABC$ is isosceles, with base BC, then $m\angle B = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle B = m\angle C$, then it is isosceles, with base BC.

Exercise 5.4: Prove Theorem 5.2 by showing that $\triangle ABC$ is congruent to its reflection $\triangle ACB$. Note that there are two parts to the theorem, and so you need to give essentially two separate arguments.

5.6. Congruence via SSS.

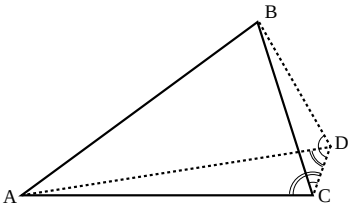
Theorem 5.3 (SSS)**.** *If $\triangle ABC$ and $\triangle A'B'C'$ are such that $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|BC| = |B'C'|$, then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. If the two triangles were not congruent, then one of the angles of $\triangle ABC$ would have measure different from the measure of the corresponding angle of $\triangle A'B'C'$. If necessary, relabel the triangles so that $\angle A$ and $\angle A'$ are two corresponding angles which differ, with $m\angle A' < m\angle A$.

We find a point D and construct the ray $\overrightarrow{AD}$ so that $m\angle DAB = m\angle A'$, and $|AD| = |A'C'|$. (That this can be done follows from Exercise 4.1) It is unclear where the point D lies: it could lie inside triangle ABC; it could lie on the line $\overleftrightarrow{BC}$ between B and C; or it could lie on the other side of the line $\overleftrightarrow{BC}$. We need to take up these three cases separately.

Exercise 5.5: Suppose the point D lies on the line $\overleftrightarrow{BC}$. Explain why this yields an immediate contradiction.

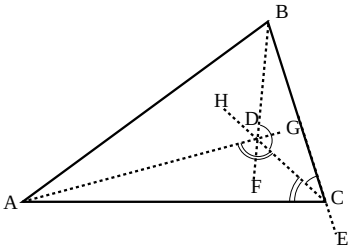
For both of the remaining cases, we draw the segments BD and CD. We observe that, by SAS, $\triangle ABD \cong \triangle A'B'C'$. It follows that $|BD| = |B'C'| = |BC|$ and that $|AD| = |A'C'| = |AC|$. Hence $\triangle BDC$ is isosceles, with base DC, and $\triangle ADC$ is isosceles with base CD. Since the base angles of an isosceles triangle have equal measure, $m\angle BDC = m\angle BCD$ and $m\angle ADC = m\angle ACD$.



First, we take up the case that D lies outside $\triangle ABC$; that is, D lies on the other side of the line BC from A.

Exercise 5.6: Finish this case of the proof, first by showing that $m\angle ADC < m\angle BDC$ and $m\angle BCD < m\angle ACD$. Then use the isosceles triangles to arrive at the contradiction that $m\angle ADC < m\angle ADC$.

We now consider the case where D lies inside $\triangle ABC$. Let E be a point on the line $\overleftrightarrow{BC}$ so that C is between B and E to some point E. Observe that $m\angle BCD + m\angle DCA + m\angle ACE = \pi$, from which it follows that $m\angle BCD + m\angle DCA < \pi$. Next, extend the line $\overleftrightarrow{BD}$ past D to some point F. Also extend the line $\overleftrightarrow{AD}$ past the point D to some point G, and extend the line $\overleftrightarrow{CD}$ past the point D to some point H.



Exercise 5.7: Finish this case of the proof by explaining why $\pi < m\angle BDC + m\angle CDA$ and $m\angle BCD + m\angle DCA < \pi$, and then show that this leads to the contradiction $\pi < \pi$.

□

5.7. Congruence via AAS.

Theorem 5.4 (AAS)**.** *Suppose we are given triangles ABC and $A'B'C'$, where $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, and $|BC| = |B'C'|$. Then $\triangle ABC \cong \triangle A'B'C'$.*

This theorem can be proved in a manner similar to the proof of Theorem 5.1 (SAS). We skip this proof here; instead, we will give a much simpler proof later, after proving the theorem about the sum of angles of a triangle.

This concludes the generalities concerning congruence of triangles. We now know the four congruence theorems, ASA, SAS, SSS and AAS. We also know that the other two possibilities, SSA and AAA, are not valid. It follows that, for example, if we are given the lengths of all three sides of a triangle, then the measures of all three angles are determined. However, we do not as yet have any means of computing the measures of these angles in terms of the lengths of the sides.

5.8. Median, altitude, and bisector in an isosceles triangle.

Definition 5.2. Two lines intersecting at a point A are perpendicular or orthogonal if each of the four angles at A has measure $\pi/2$. These angles are called right angles.

It is standard in mathematics to use the words perpendicular and orthogonal interchangeably.

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC; the median from A bisects BC (that is, it crosses BC at a point

D which is the midpoint of BC, see Exercise 3.3); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

Exercise 5.8: Show that there exists a unique median from A and a unique angle bisector at A.

Later we will show existence and uniqueness of the altitude from A (it is not quite trivial!).

In general, these three lines are distinct. However, in the isosceles triangle they coincide:

Theorem 5.5. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*

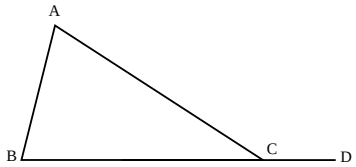
Proof. Consider triangles $\triangle ABM$ and $\triangle CBM$. Then $|AB| = |CB|$ (by definition of isosceles triangle), $|AM| = |CM|$ (by definition of midpoint), and $m\angle MAB = m\angle MCB$ (by Theorem 5.2). Thus, by SAS theorem, $\triangle ABM \cong \triangle CBM$. Therefore, $m\angle ABM = m\angle CBM$, so BM is the angle bisector.

Also, $m\angle AMB = m\angle CMB$. On the other hand, by Protractor axiom, $m\angle AMB + m\angle CMB = m\angle AMC = \pi$. Thus, $m\angle AMB = m\angle CMB = \pi/2$. □

5.9. Inequalities for general triangles.

Theorem 5.6 (Exterior angle inequality)**.** *Consider the triangle $\triangle ABC$. Let D be some point on the line BC, where C lies between B and D. Then*

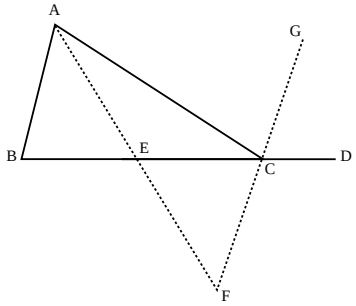
(1) $m\angle ACD > m\angle B$.
(2) $m\angle ACD > m\angle A$.



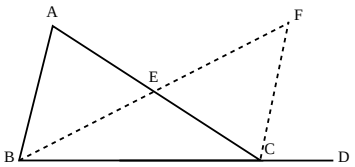
(Later we will prove a stronger result, namely, that $m\angle ACD = m\angle A + m\angle B$.)

Proof. We first prove part (1).

Choose E to be the midpoint of the segment BC, and extend AE beyond E to F, so that $|AE| = |EF|$. Now extend the line FC beyond C to some point G.



Exercise 5.9: Finish the proof of part (1) by showing that $m\angle B = m\angle BCF = m\angle DCG < m\angle DCA$ (hint: use Exercise 5.3).



Exercise 5.10: Give a proof of part (2) using the figure at left (E is the midpoint of BC, and $|EF| = |AE|$).

□

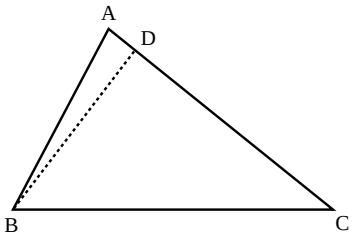
We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (Theorem 5.2). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

Theorem 5.7. *In $\triangle ABC$, if $m\angle A > m\angle B$, then we must have $|BC| > |AC|$.*

Proof. Assume not. Then either $|BC| = |AC|$ or $|BC| < |AC|$.

Exercise 5.11: Show that if $|BC| = |AC|$, the assumption $m\angle A > m\angle B$ is contradicted.

Now assume $|BC| < |AC|$, find the point D on AC so that $|BC| = |CD|$, and draw the line BD. Then $\triangle BCD$ is isosceles, with apex at C. Hence $m\angle CBD = m\angle CDB$. Since $\angle CDB$ is an exterior angle for $\triangle ABD$, by Theorem 5.6, $m\angle CDB > m\angle A$. Also, since D lies between A and C, $m\angle DBC < m\angle ABC$. We now have that $m\angle CBD < m\angle CBA < m\angle A < m\angle CDB = m\angle CBD$; so we have reached a contradiction.



□

The converse of the previous theorem is also true: opposite a long side, there must be a big angle.

Theorem 5.8. *In $\triangle ABC$, if $|BC| > |AC|$, then $m\angle A > m\angle B$.*

Proof. Assume not. If $m\angle A = m\angle B$, then $\triangle ABC$ is isosceles, with apex at C, so $|BC| = |AC|$, which contradicts our assumption.

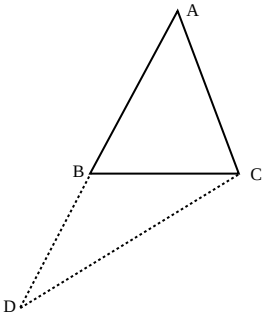
If $m\angle A < m\angle B$, then, by the previous theorem, $|BC| < |AC|$, which again contradicts our assumption. □

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 5.9 (the triangle inequality)**.** *In $\triangle ABC$, we have*

$$|AB| + |BC| > |AC|.$$

Proof. Extend the line AB past B to the point D so that $|BD| = |BC|$, and join the points C and D with a line so as to form the triangle ADC. Observe that $\triangle BCD$ is isosceles, with apex at B; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 5.7, this implies $|AD| > |AC|$. Our result now follows from the fact that $|AD| = |AB| + |BD|$ (see Theorem 3.6). □

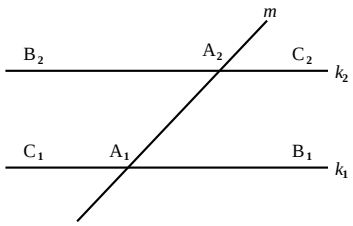


6. Parallel Lines Revisited

Looking at the proofs in the previous sections, we see that we haven't used the Parallel Lines Axiom (except for one exercise in Section 2). For example, all congruence rules for triangles were proved without using this axiom. In this section, we will see what new results can be obtained from the Parallel Lines Axiom.

6.1. Alternate interior angles. We will meet the following situation some number of times. We are given two lines k_1 and k_2, and a third line m, where m crosses k_1 at A_1 and m crosses k_2 at A_2. Choose a point $B_1 \neq A_1$ on k_1, and choose a point $B_2 \neq A_2$ on k_2, where B_1 and B_2 lie on opposite sides of the line m. Then $\angle B_1A_1A_2$ and $\angle B_2A_2A_1$ are referred to as alternate interior angles.

In any given situation, there are two distinct pairs of alternate interior angles. That is, let C_1 be some point on k_1, where B_1 and C_1 lie on opposite sides of m, and let C_2 be some point on k_2, where C_2 and B_2 lie on opposite sides of m. Then one could also regard $\angle C_1A_1A_2$ and $\angle C_2A_2A_1$ as being alternate interior angles. However, observe that $m\angle B_1A_1A_2 + m\angle C_1A_1A_2 = \pi$ and $m\angle B_2A_2A_1 + m\angle C_2A_2A_1 = \pi$. It follows that one pair of alternate interior angles are equal if and only if the other pair of alternate interior angles are equal.



Theorem 6.1. *If the alternate interior angles are equal, then the lines k_1 and k_2 are parallel.*

Proof. Suppose not. Then the lines k_1 and k_2 meet at some point D. If necessary, we interchange the roles of the B_i and the C_i so that $\angle B_1A_1A_2$ is an exterior angle of $\triangle A_1A_2D$. Then D and B_2 lie on the same side of m, so $\angle DA_2A_1 = \angle B_2A_2A_1$. By the exterior angle inequality,

$$m\angle B_1A_1A_2 > m\angle A_1A_2D = m\angle B_2A_2A_1 = m\angle B_1A_1A_2,$$

so we have reached a contradiction. □

6.2. Existence of parallel lines. Let k_1 be a line, and let A_2 be a point not on k_1. Pick some point A_1 on k_1 and draw the line m through A_1 and A_2. By Protractor axiom, we can find a line k_2 through A_2 so that the alternate interior angles are equal. Hence we can find a line through A_2 parallel to k_1.

Theorem 6.2 (alternate interior angles equal)**.** *Two lines k_1 and k_2 are parallel if and only if the alternate interior angles are equal.*

Proof. To prove the forward direction, construct the line k_3 through A_2, where there is a point B_3 on k_3, with B_3 and B_2 on the same side of m, so that $m\angle B_3A_2A_1 = m\angle B_1A_1A_2$. Then, by Theorem 6.1, k_3 is a line through A_2 parallel to k_1. Parallel Lines axiom implies $k_3 = k_1$. Hence $m\angle B_3A_2A_1 = m\angle B_2A_2A_1$, and the desired conclusion follows.

The other direction is just Theorem 6.1, restated as part of this theorem for convenience. □

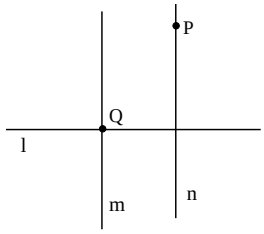
6.3. Perpendicular lines. Recall that a right angle is an angle of measure $\pi/2$, and that two intersecting lines are called perpendicular, or orthogonal, if all four angles formed by these lines are right angles (notation: $l \perp m$). Using Theorem 4.3 (about vertical and complementary angles), it is easy to see that if one of the four angles is a right angle, then so are all of them.

Proposition 6.3. *Let $m \parallel n, l \perp m$. Then $l \perp n$.*

Theorem 6.4. *For any line l and a point P, there exists a unique line n such that $P \in n, n \perp l$. This line is called the* perpendicular *from P to l.*

Proof. **Existence:** Let Q be an arbitrary point on l. By Protractor axiom, there exists a line m going through Q such that $m \perp l$. Now let n be the line going through P and parallel to m (exists by the Parallel Lines axiom). By Proposition 6.3, $n \perp l$.

Uniqueness: Assume n_1, n_2 are two lines, both containing P and perpendicular to l. Then, by Theorem 6.2, these two lines are parallel: $n_1 \parallel n_2$. But by definition, if two parallel lines have a common point, they must coincide, i.e. $n_1 = n_2$.



□

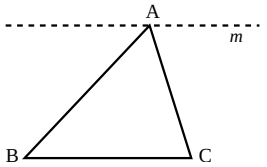
Exercise 6.1: Let A, B be distinct points and let M_1, M_2 be points on different sides of the line $\overleftrightarrow{AB}$ such that $|AM_1| = |AM_2|, |BM_1| = |BM_2|$. Show that $\overleftrightarrow{M_1M_2} \perp \overleftrightarrow{AB}$.

6.4. The sum of the angles of a triangle.

Theorem 6.5. *The sum of the measures of the angles of a triangle is equal to π.*

Proof. Consider $\triangle ABC$, and let m be the line passing through A and parallel to BC.

Exercise 6.2: Use alternate interior angles to complete the proof of this theorem.



□

Exercise 6.3: Prove that the external angle of a triangle is equal to the sum of two other angles, i.e., $m\angle ACD = m\angle A + m\angle B$ (notation as in Theorem 5.6).

Exercise 6.4: Prove Theorem 5.4 (congruence via AAS).

6.5. Parallelograms and rectangles. A quadrilateral is a figure consisting of four points A, B, C, D (vertices) and segments AB, BC, CD, DA (sides), such that all points are distinct, no three points lie on the same line, and no two sides intersect (except at vertices).

Exercise 6.5: The sum of measures of angles in a quadrilateral is equal to 2π. (Hint: cut a quadrilateral into two triangles.)

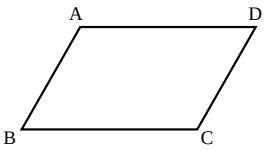
A rectangle is a quadrilateral in which all four angles are right angles. A rectangle with all four sides of equal length is a square.

Theorem 6.6. *If $ABCD$ is a rectangle, then AB is parallel to CD, and $|AB| = |CD|$. Similarly, BC is parallel to AD and $|BC| = |AD|$.*

Exercise 6.6: Prove this theorem.

(1) Prove that opposite pairs of sides are parallel.
(2) Now cut the rectangle into two triangles; prove that these two triangles are congruent. Conclude that opposite sides of the rectangle have equal length.

Somewhat more generally, a parallelogram is a quadrilateral $ABCD$ in which opposite sides are parallel; that is, AB is parallel to CD, and AD is parallel to BC.



Theorem 6.7. *Let $ABCD$ be a parallelogram. Then $m\angle A = m\angle C$; $m\angle B = m\angle D$; $|AB| = |CD|$; and $|BC| = |AD|$.*

Exercise 6.7: Prove this theorem. (Hint: Draw a diagonal.)

Theorem 6.8. *If $ABCD$ is a quadrilateral in which $|AB| = |CD|$ and $|AD| = |BC|$, then $ABCD$ is a parallelogram.*

Exercise 6.8: Prove this theorem.

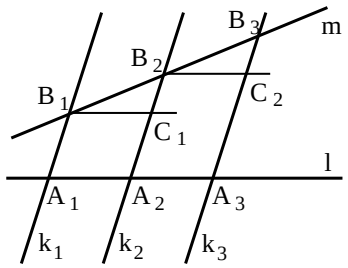
Exercise 6.9: Let $ABCD$ be a parallelogram with diagonals of equal length (that is, $|AC| = |BD|$). Then $ABCD$ is a rectangle.

The following theorem gives an important application of the results about parallelograms.

Theorem 6.9. *Let k_1, k_2, k_3 be three parallel lines. Let l be a line that intersects k_1, k_2, k_3 at points A_1, A_2, A_3 respectively and let m be another line that intersects k_1, k_2, k_3 at points B_1, B_2, B_3. Then, if $|A_1A_2| = |A_2A_3|$, then $|B_1B_2| = |B_2B_3|$.*

Proof.

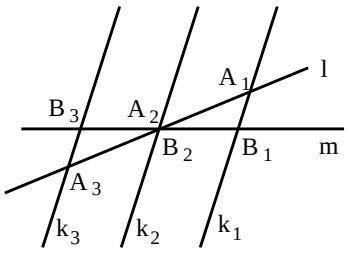
Let l_1 be a line through B_1 which is parallel to l (exists by Parallel Lines axiom) and let C_1 be the intersection point of l_1 with line k_2. Similarly, let l_2 be a line through B_2 which is parallel to l and let C_2 be the intersection point of l_2 with line k_3.



Then by construction, we have $\overleftrightarrow{B_1C_1} \| \overleftrightarrow{A_1A_2}$, and $\overleftrightarrow{A_1B_1} \| \overleftrightarrow{A_2C_1}$. Thus, $A_1B_1C_1A_2$ is a parallelogram, so by Theorem 6.7, we have $|B_1C_1| = |A_1A_2|$. In a similar way, we prove that $|B_2C_2| = |A_2A_3|$. Since $|A_1A_2| = |A_2A_3|$, we see that $|B_1C_1| = |B_2C_2|$.

Consider now triangles $\triangle B_1C_1B_2$ and $\triangle B_2C_2B_3$. We already know that $|B_1C_1| = |B_2C_2|$. In addition, $m\angle C_1B_1B_2 = m\angle C_2B_2B_3$ as alternate interior angles; similarly, $m\angle B_1B_2C_1 = m\angle B_2B_3C_2$ as alternate interior angles. By AAS congruence rule (see Theorem 6.4), we conclude that these triangles are congruent. Thus, $|B_1B_2| = |B_2B_3|$. □

Remark 6.10. As before, there are minor gaps in the proof. For example, the proof assumes that point B_2 is between B_1 and B_3 (otherwise, angles $\angle C_1B_1B_2$ and $\angle C_2B_2B_3$ would not be alternate interior angles) and that all points used in the proof are distinct (for example, that $A_1 \neq B_1$). Clearly this is so in the figure we used, but the figure shows only one of the many possible configurations; for example, the configuration shown below (in which $A_2 = B_2$) also satisfies all the assumptions of the theorem. With a little bit of extra work, one can fill these gaps and make the proof which works for all possible configurations.



7. Similarity and Pythagoras Theorem

7.1. Similar triangles. We say that triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar with constant of proportionality k if $\angle A = \angle A', \angle B = \angle B', \angle C = \angle C'$ and

$$\frac{|A'B'|}{|AB|} = \frac{|B'C'|}{|BC|} = \frac{|A'C'|}{|AC|} = k.$$

We will use notation $\triangle ABC \sim \triangle A'B'C'$.

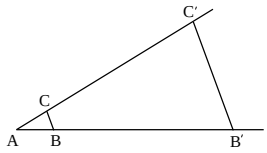
From this definition, it is clear that $\triangle ABC \cong \triangle A'B'C'$ iff they are similar with constant of proportionality $k = 1$.

Exercise 7.1: Show that if $\triangle ABC \sim \triangle A'B'C'$ with constant k_1 and $\triangle A'B'C' \sim \triangle A''B''C''$ with constant k_2, then $\triangle ABC \sim \triangle A''B''C''$ with constant k_1k_2.

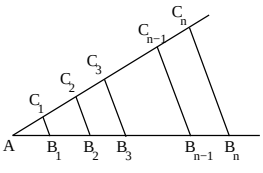
7.2. Key theorem. The key tool in the study of similar triangles is the following theorem.

Theorem 7.1. *Consider a triangle $\triangle ABC$ and let $B' \in \overrightarrow{AB}$, $C' \in \overrightarrow{AC}$ be such that lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel. Then*

$$\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}$$



Proof. The proof of this theorem is unexpectedly difficult. We will only give a proof in the case when $\frac{|AB'|}{|AB|} = n$ is an integer. The general case requires some calculus and is not discussed here.



So, assume $\frac{|AB'|}{|AB|} = n$. Divide the segment AB' into n equal length pieces, i.e. find on it points $B_1 = B, B_2, \ldots, B_n = B'$ such that $|AB_1| = |B_1B_2| = \cdots = |B_{n-1}B_n|$. Through each point B_i, draw a line k_i which is parallel to $\overleftrightarrow{BC}$. Let C_i be the intersection point of k_i with $\overrightarrow{AC}$, so $C_1 = C$, $C_n = C'$.

Then by Theorem 6.9, we have $|C_1C_2| = |AC_1|$, $|C_2C_3| = |C_1C_2|$, and so on. Thus, for every i we get $|C_iC_{i+1}| = |AC|$, so $|AC'| = n|AC|$, and

$$\frac{|AC'|}{|AC|} = n = \frac{|AB'|}{|AB|}$$

□

Exercise 7.2: Show that conversely, if $B' \in \overrightarrow{AB}, C' \in \overrightarrow{AC}$ are such that $\frac{|AC'|}{|AC|} = \frac{|AB'|}{|AB|}$, then $\overleftrightarrow{B'C'} \| \overleftrightarrow{BC}$.

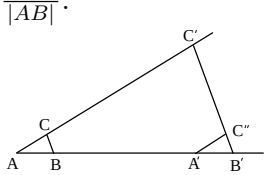
Exercise 7.3: Prove Theorem 7.1 in the case when $\frac{|AB'|}{|AB|} = \frac{m}{n}$ is a rational number.

7.3. Existence of similar triangles.

Theorem 7.2. *Using the same notation as in Theorem* 7.1, *$\triangle ABC \sim \triangle AB'C'$.*

Proof. By Theorem 6.2 (alternate interior angles equal), $\angle B = \angle B'$ and $\angle C = \angle C'$. By Theorem 7.1, $\frac{|AC'|}{|AC|} = \frac{|AB'|}{|AB|}$. Thus, it remains to show that $\frac{|BC'|}{|BC|} = \frac{|AB'|}{|AB|}$.

Let A' be a point on $\overrightarrow{BA}$ such that $|A'B'| = |AB|$, and let $C'' \in \overrightarrow{BC}$ be such that $\overleftrightarrow{A'C''} \| \overleftrightarrow{AC'}$.



Exercise 7.4: Show that $\triangle A'B'C'' \cong \triangle ABC$.

Using Theorem 7.1, one easily sees that $\frac{|B'C'|}{|B'C''|} = \frac{|AB'|}{|A'B'|}$. Since $|A'B'| = |AB|$, and $|B'C''| = |BC|$, we get $\frac{|B'C'|}{|BC|} = \frac{|AB'|}{|AB|}$. □

Corollary 7.3. *For any triangle $\triangle ABC$ and a real number $k > 0$, there exists a triangle $\triangle A'B'C'$ similar to $\triangle ABC$ with constant k.*

Exercise 7.5: For a triangle $\triangle ABC$, let D be the midpoint of AB and F be the midpoint of AC. Show that

(1) $\overleftrightarrow{DF} \| \overleftrightarrow{BC}$
(2) $|DF| = \frac{1}{2}|BC|$

7.4. Similarity via AAA.

Theorem 7.4 (Similarity via AAA)**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that $\angle A = \angle A', \angle B = \angle B', \angle C = \angle C'$. Then these triangles are similar.*

Proof. Let $k = \frac{|A'B'|}{|AB|}$. Construct a triangle $\triangle A''B''C''$ which is similar to $\triangle ABC$ with constant of proportionality k. Then $|A'B'| = |A''B''|$, and $\angle A = \angle A' = \angle A''$, $\angle B = \angle B' = \angle B''$, $\angle C = \angle C' = \angle C''$. Thus, by ASA, $\triangle A'B'C' \cong \triangle A''B''C''$. □

Theorem 7.5 (Similarity via SAS)**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that $\angle A = \angle A'$, $\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}$. Then these triangles are similar.*

Exercise 7.6: Prove this theorem.

7.5. Pythagoras Theorem. A right triangle is a triangle in which one of the angles is a right angle. A hypotenuse is the side opposing the right angle.

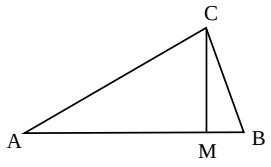
The following theorem is one of the oldest known to humanity. According to some studies, this result may have been known as early as 4 thousand years ago — long before Pythagoras or Euclid. There is a number of ways to prove this theorem. The proof given below is not the most geometrically intuitive — but it is the easiest one to derive from the axioms.

Theorem 7.6 (Pythagoras Theorem)**.** *Let $\triangle ABC$ be a right triangle, with $\angle C$ being the right angle. Then*

$$|AB|^2 = |AC|^2 + |BC|^2.$$

Proof. Denote for brevity $a = |BC|, b = |AC|, c = |AB|$. Drop a perpendicular from C to AB; let M be the point where this perpendicular intersects AB.

Exercise 7.7: Show that $\triangle ACM \sim \triangle ABC$, and deduce from this that $|AM| = b^2/c$.



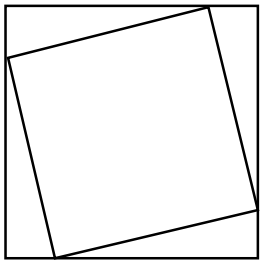
Exercise 7.8: Show that $\triangle CBM \sim \triangle ABC$, and deduce from this that $|BM| = a^2/c$.

Combining these two exercises, we get

$$c = |AM| + |MB| = \frac{a^2}{c} + \frac{b^2}{c}.$$

Multiplying both sides by c, we get the Pythagoras theorem. □

Exercise 7.9: The figure to the right can be used to give a more "geometrically obvious" proof of Pythagoras theorem — if we allow ourselves to use the notion of "area".



(1) By computing the area of the large triangle in two ways, prove the Pythagoras theorem.
(2) Carefully analyze the proof of part (1) and list all the properties of area you are using. Can you prove any of them? (This, of course, depends on how one defines area.)

Exercise 7.10: Let $\triangle ABC$ and $\triangle A'B'C'$ be such that $|AB| = |A'B'|$, $|BC| = |B'C'|$, and $m\angle C = m\angle C' = \pi/2$. Prove that $\triangle ABC \cong \triangle A'B'C'$.

8. Circles and lines

8.1. Circles. A circle Σ is the set of points at fixed distance $r > 0$ from a given point, its center. The distance r is called the radius of the circle Σ.

The circle Σ divides the plane into two regions: the inside, which is the set of points at distance less than r from the center O, and the outside, which consists of all points having distance from O greater than r. Note that every line segment from O to a point on Σ has the same length r.

A line segment from O to a point on Σ is also called a radius; this should cause no confusion.

A line segment connecting two points of Σ is called a chord, if the chord passes through the center, then it is called a diameter.

As above, we also use the word diameter to denote the length of a diameter of Σ, that is, the number that is twice the radius.

8.2. Perpendicular bisector. Let A, B be distinct points. The perpendicular bisector of segment AB is the line l which contains midpoint of AB and is perpendicualr to $\overleftrightarrow{AB}$.

Theorem 8.1. *Let A, B be distinct points. Then $|OA| = |OB|$ iff O lies on the perpendicaurl bisector of AB.*

Corollary 8.2. *If A, B are two distinct points on a circle Σ, then the center of Σ lies on perpendicular bisector of AB.*

Proposition 8.3. *A line k intersects a circle Σ in at most two points.*

Exercise 8.1: Prove this proposition, using proof by contradiction.

8.3. Circumscribed circles. The circle Σ is circumscribed about $\triangle ABC$ if all three vertices of the triangle lie on the circle. In this case, we also say that the triangle is inscribed in the circle.

Note that another way to describe a circle circumscribed about a triangle is to say that it is the smallest circle for which every point inside the triangle is also inside the circle. In this view, the problem of circumscribing a circle becomes a minimization problem. A given triangle lies inside many circles, but the circumscribed circle is, in some sense, the smallest circle which lies outside the given triangle.

It is not immediately obvious that one can always solve this minimization problem, nor that the solution is unique.

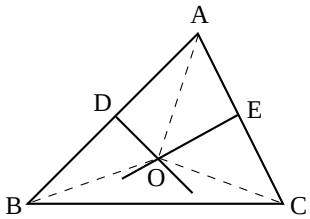
Proposition 8.4 (Uniqueness of Circumscribed Circles)**.** *There is at most one circle circumscribed about any triangle.*

Proof. Suppose there are two circles Σ and Σ' which are circumscribed about $\triangle ABC$. Since points A, B, and C lie on both circles, AB and BC are chords. By Corollary 8.2, the perpendicular bisectors of AB and BC both pass through the centers of Σ and Σ'. Since these two distinct lines can intersect in at most one point, Σ and Σ' share the same center O. Since AO is a radius for both circles, they have the same center and radius, and hence are the same circle. □

Theorem 8.5 (Existence of Circumscribed Circles)**.** *Given $\triangle ABC$, there is always exactly one circle Σ circumscribed about it.*

Proof. We need to show existence of a circumscribed circle; uniqueness was shown in Proposition 8.4.

Let D and E be the midpoints of sides AB and BC respectively. Draw the perpendicular bisectors of AB and BC, and let O be the point where these two lines intersect (note that O need not be inside the triangle). Draw the lines AO, BO and CO. By Theorem 8.1, $|AO| = |BO|$ (since O lies on the perpendicular bisector of AB); similarly, $|BO| = |CO|$. Thus, if we denote $r = |AO| = |BO| = |CO|$, and let Σ be the circle with center at O and radius r, then points A, B, C are on Σ.



□

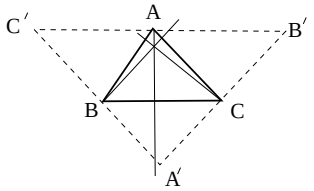
Corollary 8.6. *In any triangle, the three perpendicular bisectors of the sides meet at a point.*

Exercise 8.2: Explain why Theorem 8.5 implies this corollary.

8.4. Altitudes meet at a point.

Theorem 8.7. *In any triangle $\triangle ABC$, the three altitudes meet at a point.*

Proof. Draw line l through vertex A, such that $l \parallel \overleftrightarrow{BC}$; similarly, draw lines through vertices B and C parallel to opposite sides of $\triangle ABC$. Let A', B', C' be the intersection points of these lines, as shown in the figure.



Exercise 8.3:
(1) Prove that each of triangles $\triangle A'BC, \triangle ABC', \triangle AB'C$ is congruent to $\triangle ABC$.
(2) Prove that A is the midpoint of $B'C'$, B is the midpoint of $A'C'$, and C is the midpoint of $A'B'$.
(3) Prove that altitudes of $\triangle ABC$ are the same as perpendicular bisectors of sides of $\triangle A'B'C'$.

Since, by Corollary 8.6, perpendicular bisectors of $\triangle A'B'C'$ meet at a point, we see that altitudes of $\triangle ABC$ meet at a point. □

8.5. Tangent lines. A line that meets a circle in exactly one point is a tangent line to the circle at the point of intersection. Our first problem is to show that there is one and only one tangent line at each point of a circle.

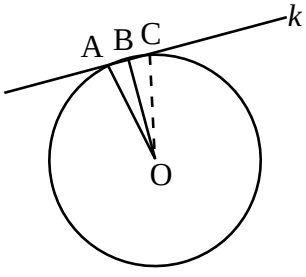
Proposition 8.8. *Let A be a point on the circle Σ, and let k be the line through A perpendicular to the radius at A. Then k is tangent to Σ.*

Proof. There are only three possibilities for k: it either is disjoint from Σ, which cannot be, as A is a common point; or it is tangent to Σ at A; or it meets Σ at another point B. If k meets Σ at B then OAB is a triangle, where $\angle A$ is a right angle. Since OA and OB are both radii, $|OA| = |OB|$. Hence $\triangle OAB$ is isosceles. Hence $m\angle A = m\angle B$. We have constructed a triangle with two right angles, which cannot be; i.e., we have reached a contradiction. □

Proposition 8.9. *If k is a line tangent to the circle Σ at the point A, then k is perpendicular to the radius ending at A.*

Proof. We will prove the contrapositive: if k is a line passing through A, where k is not perpendicular to the radius, then k is not tangent to Σ.

Draw the line segment m from O to k, where m is perpendicular to k. Let B be the point of intersection of k and m. On k, mark off the distance $|AB|$ from B to some point C, on the other side of B from A. Since OB is perpendicular to k, $m\angle OBA = m\angle OBC$. By SAS, $\triangle OBA \cong \triangle OBC$, and so $|OC| = |OA|$. Thus both A and C lie on Σ, and k intersects Σ in two points. Thus, k is not tangent to Σ.



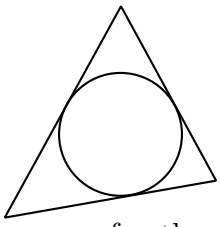
□

Corollary 8.10. *Let A be a point on the circle Σ. Then there is exactly one line through A tangent to Σ.*

Exercise 8.4: Prove this Corollary.

8.6. Inscribed circles.

A circle Σ is inscribed in $\triangle ABC$ if all three sides of the triangle are tangent to Σ. One can view the inscribed circle as being the largest circle whose interior lies entirely inside the triangle. (Note that it is not quite correct to say that the circle lies entirely inside the triangle, because the triangle and the circle share three points.)



We start the search for the inscribed circle with the question of what it means for the circle to have two tangents which are not parallel.

Proposition 8.11. *Let A be a point outside the circle Σ, and let k_1 and k_2 be tangents to Σ passing through A. Then the line segment OA bisects the angle between k_1 and k_2.*

Proof. Let B_i be the point where k_i is tangent to Σ, for $i = 1, 2$. Draw the lines OB_1 and OB_2. Observe that $|OB_1| = |OB_2| = r$, and that, since radii are perpendicular to tangents, $\angle OB_1A = \angle OB_2A = \pi/2$. By Pythagoras theorem, $|AB_1| = \sqrt{|AB_1|^2 + r^2} = |AB_2|$.

By SSS, $\triangle OB_1A \cong \triangle OB_2A$. Hence $m\angle OAB_1 = m\angle OAB_2$. □

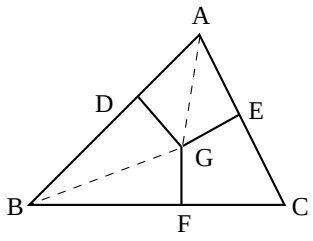
From the above, we see that if there is an inscribed circle for $\triangle ABC$, then its center lies at the point of intersection of the three angle bisectors, and its radius is the distance from this point to the three sides. Hence we have proven the following.

Corollary 8.12 (Inscribed circles are unique)**.** *Every triangle has at most one inscribed circle.*

Theorem 8.13. *Every triangle has an inscribed circle.*

Proof.

Let G be the point of intersection of the angle bisectors from A and B in $\triangle ABC$. Let D be the point where the perpendicular from G meets AB; let E be the point where the perpendicular from G meets BC; and let F be the point where the perpendicular from G meets AC.



Observe that, by AAS, $\triangle ADG \cong \triangle AFG$. Similarly, $\triangle BDG \cong \triangle BEG$ and $\triangle CEG \cong \triangle CFG$.

We have shown that the perpendiculars from G to the three sides all have equal length; call this length r. Then, by Proposition 8.8, the circle centered at G of radius r is tangent to the three sides of $\triangle ABC$ exactly at the points D, E and F. □

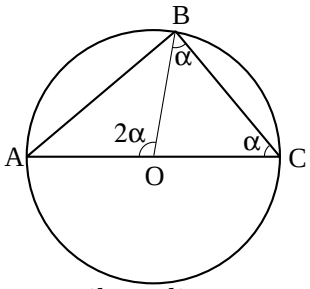
Corollary 8.14. *The three angle bisectors of a triangle meet at a point; this point is the center of the inscribed circle.*

Exercise 8.5: Give a proof of this corollary using the above theorem.

Exercise 8.6: Let A and B be points on the circle Σ. Let k be the line tangent to Σ at A and let m be the line tangent to Σ at B. Prove that if k and m are parallel, then the line segment AB is a diameter of Σ.

8.7. CENTRAL ANGLES. Let Σ be a circle with center O, and let A, B be points on Σ. Then the angle $\angle AOB$ is called central angle. It turns out that the angles in a triangel ABC inscribed in Σ are closely related with the corresponding central angles.

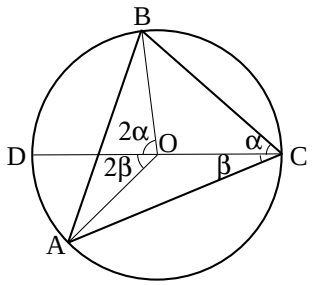
Proposition 8.15. *Let Σ be a circle with center O, and let A, B, C be distinct points on Σ such that AC is a diameter of Σ. Then $m\angle ACB = \frac{1}{2}m\angle AOB$*



Proof. Consider the triangle BOC. Since $|BO| = |OC|$, this triangle is isosceles. Thus, by Theorem 5.2(base angles are equal), $\angle OBC = \angle OCB$. Now consider $\angle AOC$. This is an external angle of $\triangle OBC$, so by Exercise 6.3, it is equal to the sum of two other angles: $\angle AOC = \angle OBC + \angle OCB = 2\angle OCB = 2\angle ACB$. □

The next step is to generalize it to the case when AC is not necessarily a diameter of Σ. however, one must be careful when doing this. The following "theorem" seems a natural generalization — however, it is not correct as stated. We give it here as an example of why it is dangerous to base your proof on things which are "obvious from the figure".

Theorem 8.16 (INCORRECT)**.** *Let Σ be a circle with center O, and let A, B, C be distinct points on Σ. Then $m\angle ACB = \frac{1}{2}m\angle AOB$.*

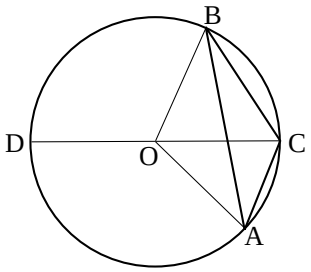


"Proof". Let D be the point on Σ such that CD is a diameter (it is easy to show that such a point exists and is unique). Then $m\angle ACB = m\angle ACD + m\angle DCB$. Since CD is a diameter, we can apply Proposition 8.15 to triangles $\triangle ACD, \triangle DCB$ which gives $\angle ACD = \frac{1}{2}\angle AOD, m\angle DCB = \frac{1}{2}m\angle DOB$, so

$$m\angle ACB = \frac{1}{2}(m\angle AOD + m\angle DOB) = \frac{1}{2}m\angle AOB$$

□

So what is wrong with this theorem and this proof? Here is one problem: if we choose A, B, C so that $\angle ACB > \pi/2$ as shown below, then according to this theorem, $\angle AOB = 2\angle ACB > \pi$. But by Protractor axiom, the measure of any angle is $\leq \pi$. So we get a contradiction which shows that this theorem can not be correct as stated.



Closer look also shows what is the likely origin of this trouble. Namely, looking at this example it seems that the formula $m\angle ACB = \frac{1}{2}m\angle AOB$ would be true if we gave different interpretation of $m\angle AOB$: if instead of measuring the smaller of two angles formed by rays OA and OB (which is the definition we used in Protractor axiom and elsewhere), we measured that of the two angles which contains the point D. This also shows the gap in the proof: the proof assumes that $m\angle AOD + m\angle DOB = m\angle AOB$; however, we didn't explain why it is so. It could be justified by referring to Protractor axiom — but only if the ray $\overrightarrow{OD}$ is inside angle $\angle AOB$. As the two figures above show, this is not always true.

As mentioned above, the statement of the theorem can be corrected. There are several ways of doing so. One possibility is to change the way we measure angles, so instead of saying "for every angle we have its measure", we would say "for every sector there is a measure", with a sector being one of two regions of the plane bounded by the angle. Then replacing

in Theorem 8.16 $m\angle AOB$ by "measure of the sector bounded by $\angle AOB$ which does not contain point C" would give a correct theorem.

This can be done (and, in fact, this is the way it is done in most elementary geometry books), but it would require some work — and it is too late to do so now, as we have already extensively used the notion of anlge and Protractor axiom. Therefore, instead we give the following reformulation of Theorem 8.16.

Theorem 8.17. *Let Σ be a circle with center O, and let A, B, C be distinct points on Σ. Then*

$$m\angle AOB = \begin{cases} 2m\angle ACB, & \text{if } m\angle ACB \leq \pi/2 \\ 2\pi - 2m\angle ACB, & \text{if } m\angle ACB > \pi/2 \end{cases}$$

9. Coordinates

In this section, we show how one can relate this axiomatic approach to Euclidean geometry with the familiar coordinate one, in which we use a coordinate system to describe a point by a pair of real numbers — its x and y coordinates. Please note that this is a relatively new approach to geometry: it was introduced Descartes in 17th century — less than 4 centuries ago (for comparison, Euclid's *Elements* were written 23 centuries ago). We will discuss advantages and disadvantages of this approach later.

9.1. Coordinate system. A coordinate system is an identification $f\colon P \to \mathbb{R}^2$, where P is the plane (i.e., the set of all point considered in Euclidean geometry) and $\mathbb{R}^2$ is the set of all pairs (x, y) of real numbers. This naturally extends the notion of coordinate system on a line, discussed in Ruler Axiom.

As with a line, there is more than one coordinate system on the plane. In order to define a coordinate system, we need to specify the origin and coordinate axes. Here are the precise definitions.

Definition 9.1. A coordinate system on the plane is the following collection of data:

- A point O (called the origin).
- Rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ such that $\overleftrightarrow{OA} \perp \overleftrightarrow{OB}$.

The lines OA and OB are usually called x-axis and y-axis respectively. Please note that the definition of coordinate system asks not just for the lines but for the rays — this is needed to determine the direction on each of the axes.

Now comes the promised result about identifying the set of all points with $\mathbb{R}^2$.

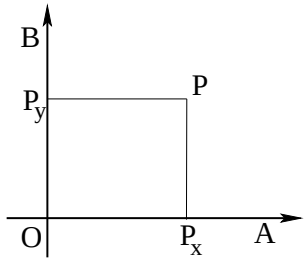
Theorem 9.1. *Every coordinate system $O, \overrightarrow{OA}, \overrightarrow{OB}$ defines an identification of the set of all points with $\mathbb{R}^2$.*

Proof. To define an identification, we need:

- Describe a map $f\colon \{points\} \to \mathbb{R}^2$
- Show that conversely, for each $(x, y) \in \mathbb{R}^2$, there is a unique point P corresponding to it (i.e., such that $f(P) = (x, y)$).

To define f, note first that by Ruler Axiom, choice of O and a ray $\overrightarrow{OA}$ defines a coordinate system $f_x\colon \overleftrightarrow{OA} \to \mathbb{R}$ such that $f_x(O) = 0, f_x(A) > 0$. Similarly, ray $\overrightarrow{OB}$ defines a coordinate system $f_y\colon \overrightarrow{OB} \to \mathbb{R}$. This allows us to label points on both axes by real numbers.

Now let P a point. Drop perpendiculars PP_x, PP_y from P to $\overleftrightarrow{OA}$ (x-axis) and $\overleftrightarrow{OB}$ (y-axis) (such perpendiculars exist and are unique by Theorem 6.4). So we have two "projections" of P on the axes. Next, define the x and y coordinates $x = f_x(P_x), y = f_y(P_y)$ by using the coordinate systems f_x on the x-axis and f_y on the y-axis. Thus, we have defined a map which for a given point P gives pair of real numbers x and y. We will say that x, y are coordinates of P, or that P has coordinates x, y.



Conversely, let x, y be real numbers. To show that there is a unique point P with coordinates x, y, let P_x be the point on the x-axis such that $f_x(P_x) = x$ (such a point exists and is unique by the Ruler Axiom); similarly, let P_y be the point on y-axis such that $f_y(P_y) = y$. Let l be the perpendicular to x-axis through P_x (exists by Protractor Axiom), and m the perpendicular to y-axis through P_y. Let P be the intersection point of l and m. Then we

claim that P has coordinates (x, y) we started with, and moreover, P is the only point that has these coordinates. The proofs of these two statements is left as an easy exercise to the reader. □

As usual, we will write $P = (x, y)$ to say "point P has coordinates (x, y)". We will also commonly use word "horizontal" for a line which is parallel to x-axis and "vertical" for a line which is parallel to y-axis.

Exercise 9.1: Show that any horizontal line is perpendicular to any vertical line.

Exercise 9.2: Show that two distinct points A, B have the same coordinate iff $\overleftrightarrow{AB}$ is a vertical line.

9.2. Equation of a line. In this section we will show that any line l not parallel to y axis can be described by an equation $y = mx + b$. This is not quite easy and requires some preparation. Throughout this section, we assume that we have chosen some coordinate system on the plane.

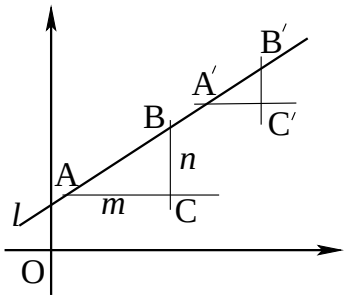
Exercise 9.3: Let $A = (x_1, y_1), B = (x_2, y_2)$ be distinct points. Prove that $\overleftrightarrow{AB}$ is parallel to the y-axis iff $x_1 = x_2$.

Definition 9.2. Let $A = (x_1, y_1), B = (x_2, y_2)$ be points such that $x_1 \neq x_2$. Then we define slope of segment AB by

$$m(AB) = \frac{y_2 - y_1}{x_2 - x_1}$$

Theorem 9.2. *Let l be a line which is not parallel to the y-axis, and let A, B, A', B' be points on l such that $A \neq B, A' \neq B'$. Then the slopes of segments AB and $A'B'$ are equal: $m(AB) = m(A'B')$.*

Proof.



Let m be the line through A parallel to x-axis (exists and is unique by Parallel lines axiom), and n the line through B parallel to y-axis. By Exercise 9.1, $m \perp n$. Let C be the intersection point of m, n. Then $\triangle ABC$ is the right triangle: $m\angle C = \pi/2$, and $|AC| = x_2 - x_1$, $|BC| = y_2 - y_1$ where $A = (x_1, y_1), B = (x_2, y_2)$.
Similarly, let m' be the line through A' parallel to x-axis, and n' the line through B' parallel to y-axis, and let C' be the intersection point of m', n'. Then $\triangle A'B'C'$ is the right triangle: $m\angle C' = \pi/2$, and $|A'C'| = x_2' - x_1'$, $|B'C'| = y_2' - y_1'$ where $A' = (x_1', y_1'), B' = (x_2', y_2')$.

Using Theorem 6.2, we see that $m\angle A = m\angle A', m\angle B = m\angle B'$. Thus, $\triangle ABC \sim \triangle A'B'C'$ by AAA. Thus, by definition of similar triangles, $\frac{|A'C'|}{|AC|} = \frac{|B'C'|}{|BC|}$. Denoting this ratio by k, we get $x_2' - x_1' = k(x_2 - x_1), y_2' - y_1' = k(y_2 - y_1)$, so

$$\frac{y_2' - y_1'}{x_2' - x_1'} = \frac{y_2 - y_1}{x_2 - x_1}$$

□

Exercise 9.4: This proof actually has the same deficiencies as our (incorrect) proof of the theorem about central angles: it uses some information about relative positions of points on the line l which is true in the figure shown but was not proved (and, in fact, may be false) in general. Can you identify what information it uses and in which step?

Fortunately, the theorem is still true: even though the proof above has gaps, it can be fixed. Can you do this?

This theorem implies that for a line l not parallel to y-axis, we can define its slope $m(l)$ as the slope of any segment on this line. According to the theorem above, the result doesn't depend on which segment we used.

Now we are ready to prove the main result about equation of a line.

Theorem 9.3. *Let l be a line with slope m which contains point $P = (x_0, y_0)$. Then a point $A = (x, y)$ lies on l iff x, y satisfy the equation*

$$y - y_0 = m(x - x_0)$$

Proof. First, we prove that if $A \in l$ then x, y satisfy this equation. Indeed, by Theorem 9.2 and the definition of the slope of a line, the slope of AP must be equal to the slope of l, so $\frac{y-y_0}{x-x_0} = m$. This is equivalent to the equation above.

Conversely, assume that x, y satisfy $y - y_0 = m(x - x_0)$. We need to prove that $A \in l$.

Consider the line going through A and parallel to y-axis. Let $A' = (x', y')$ be the point of intersection of this line with l. Since $\overleftrightarrow{AA'}$ is parallel to y-axis, points A and A' have the same x-coordinate. Thus, $x = x'$. Next, by previous argument, $y' - y_0 = m(x' - x_0) = m(x - x_0)$. Thus, $y' = m(x - x_0) + y_0 = y$. So $A = A'$. Since by construction '$A' \in l$, this gives $A \in l$. □

Of course, writing the equation of a line is only the beginning. We could continue in this vein and develop equations of a circle, develop trigonometry and so on. However, as we do not have time to cover all this (and most of this you have already seen in other courses), we stop here.

9.3. Advantages and disadvantages of coordinate method. One of the natural questions people ask after seeing the coordinate method is this: why don't we just forget axiomatic approach to Euclidean geometry and start by defining the plane to be the set $\mathbb{R}^2$, let lines be defined by equations like $y = mx+b$, and so on? In fact, some mathematicians (for example, French mathematician J. Dieudonne) have suggested this approach to the study of geometry. However, this has some serious drawbacks. For example, consider Corollary 8.14: three angle bisectors in a triangle intersect at a single point. The proof given in these notes (and going back to Euclid) is rather nice and is based essentially on the fact that there is a unique inscribed circle. However, proving the same theorem using the coordinate approach, by writing equations of the three angle bisectors and then showing that these three equations have a common solution, while not impossible, results in 2 pages of extremely messy computations. So the coordinate approach, while powerful, is not a replacement for a more traditional approach: the best way would to to combine them. By the way, Descartes himself was fully aware of the drawbacks of the coordinate approach and never suggested that it is a is a magical cure-all.

And for the purposes of MAT 200, we certainly want the axiomatic approach: the whole point of this part of the course was to show you logic in action, proving results starting with the axioms and advancing to more and more complicated ones. Axiomatic approach to Euclidean geometry provides a very good example of this.