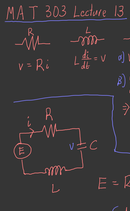
MAT 303 Lecture 13



the gizmos having batteries

$$E = Ri + v + L\frac{di}{dt} \qquad C\frac{dv}{dt} = i$$

$$\Rightarrow \begin{cases} \frac{di}{dt} = \frac{-R}{L}i - \frac{1}{L}v + \frac{1}{L}E \\ \frac{dv}{dt} = \frac{1}{C}i \end{cases}$$

Kirchhoff's Current Law:
conservation of current at nodes
sum of currents going in is the sum of currents going out

α $i_1 + i_2 + i_3 = 0$

β $v - Ri_2 = 0$

γ $Ri_2 - L\frac{di_3}{dt} = 0$

δ $C\frac{dv}{dt} = i_1$

(both nodes give the same equation)

Because there are derivatives on i_3 and v, and as we have two equations with no derivatives, we will pass to a 2x2 system purely in terms of i_3 and v

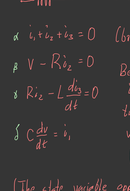
(The state variable approach)

$$C\frac{dv}{dt} \overset{\delta}{=} i_1 \overset{\alpha}{=} -i_2 - i_3 \overset{\beta}{=} -\frac{1}{R}v - i_3$$

$$L\frac{di_3}{dt} \overset{\gamma}{=} Ri_2 \overset{\beta}{=} v$$

$$\begin{bmatrix} v' \\ i_3' \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC} & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v \\ i_3 \end{bmatrix}$$

this is a familiar first order linear, constant homogeneous system, yay!



α $i_1 = i_2 + i_3$

β $E = L_1\frac{di_1}{dt} + v$

γ $0 = L_2\frac{di_2}{dt} + Ri_3 - v$

δ $C\frac{dv}{dt} = i_2$

$$\frac{di_1}{dt} = \frac{-1}{L_1}v + \frac{1}{L_1}E(t)$$

$$\frac{di_2}{dt} = \frac{-R}{L_2}i_3 + \frac{1}{L_2}v$$

$$\frac{dv}{dt} = \frac{1}{C}i_1 - \frac{1}{C}i_3$$

$$\begin{bmatrix} i_1' \\ i_2' \\ i_3' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1/L_1 \\ 0 & -R/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_3 \\ v \end{bmatrix} + \begin{bmatrix} E(t)/L_1 \\ 0 \\ 0 \end{bmatrix}$$

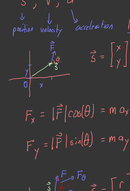
Dynamics

(I) Vector analysis primer (planar case)

$\vec{s}$, $\vec{v}$, $\vec{a}$ — position, velocity, acceleration

$\vec{s}' = \vec{v}$ $\quad \vec{v}' = \vec{a}$ — Only valid in rect. coordinates!

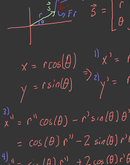
Newton's Second Law: $\vec{F} = m\vec{a}$



$\vec{s} = \begin{bmatrix} x \\ y \end{bmatrix}$ (Rectangular coordinates)

$$F_x = |\vec{F}|\cos(\theta) = ma_x = mx''$$

$$F_y = |\vec{F}|\sin(\theta) = ma_y = my''$$



$\vec{s} = \begin{bmatrix} r \\ \theta \end{bmatrix}$ (Polar coordinates)

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

1) $x' = r'\cos(\theta) - r\sin(\theta)\theta'$
2) $y' = r'\sin(\theta) + r\cos(\theta)\theta'$ $\Rightarrow$

3) $x'' = r''\cos(\theta) - r'\sin(\theta)\theta' - r'\sin(\theta)\theta' - r\cos(\theta)\theta'^2 - r\sin(\theta)\theta''$
$= \cos(\theta)r'' - 2\sin(\theta)r'\theta' - r\cos(\theta)(\theta')^2 - r\sin(\theta)\theta''$

4) $y'' = \sin(\theta)r'' + 2\cos(\theta)r'\theta' - r\sin(\theta)(\theta')^2 + r\cos(\theta)\theta''$

Suppose $\underline{\theta = 0}$ for a second, then $r = x$

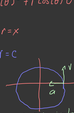
$$v_r = x' \qquad a_r = x''$$

$$v_\theta = y' \qquad a_\theta = y''$$

1) $v_r = r'$ $\quad$ 2) $v_\theta = r\theta'$

3) $a_r = r'' - r(\theta')^2$ $\quad$ 4) $a_\theta = 2r'\theta' + r\theta''$

$r = x$



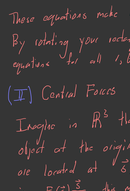
For centripetal circular motion, $r' = 0$, but $\theta' \neq 0$, in fact, $a_r = -r(\theta')^2 = \frac{-v_\theta^2}{r}$

These equations make no reference to x, y
By rotating your rectangular coordinates get these equations for all r, θ! (not just $\theta = 0$)

(II) Central Forces

Imagine in $\mathbb{R}^3$ that there is some attracting object at the origin. This means that if you are located at $\vec{s}$, the force acting on you is $F(\vec{s})\frac{\vec{s}}{|\vec{s}|}$, this means that there is a pull of magnitude $F(\vec{s})$ directed away from the origin

Example:
electric repulsion $\rightarrow$ $q_1 q_2 r^{-2}$
gravitational attraction $\leftarrow$ $m_1 m_2 r^{-2}$



(1) $mx'' = F\frac{x}{r}$
(2) $my'' = F\frac{y}{r}$
(3) $mz'' = F\frac{z}{r}$

$-y(1) + x(2)$:

$$mxy'' - myx'' = 0 \Rightarrow xy'' - yx'' = 0$$

similarly:

$$yz'' - zy'' = 0 \qquad zx'' - xz'' = 0$$

$$\int(xy'' - yx'')dt = xy' - yx' - \cancel{\int(x'y' - y'x')dt}^{\,0}$$

$$\Rightarrow xy' - yx' = C_3$$

similarly:

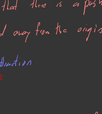
$$yz' - zy' = C_1 \qquad zx' - xz' = C_2$$

$$C_1 x + C_2 y + C_3 z$$

$$= xyz' - xzy' + yzx' - yxz' + zxy' - zyx'$$

$$= 0$$

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$



equation of a plane through the origin! ∎