MAT 126: PROBLEM SET 2 SOLUTIONS

NOTE: *I am not going to typeset the process by which I solved the systems of equations arising in partial fractions problems. However, your answers should include this for the sake of being complete.*

0. (a)

$$\int \left[x^5 + 3x^2 + x + 7\right] dx = \frac{1}{6}x^6 + x^3 + 7x + C. \qquad \square$$

0. (b)

$$\int \cos(2x)\,dx \left[\begin{array}{l}\text{Let } u = 2x\\ du = 2\,dx\end{array}\right] = \frac{1}{2}\int \cos(u)\,du$$

$$= \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(2x) + C \qquad \square$$

0. (c)

$$\int xe^x\,dx \left[\begin{array}{ll} u = x & dv = e^x\,dx\\ du = dx & v = e^x\end{array}\right] = xe^x - \int e^x\,dx$$

$$= xe^2 - e^x + C \qquad \square$$

0. (d)

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$= \frac{A(x-1) + B(x+1)}{(x+1)(x-1)} = \frac{(A+B)x + (B-A)}{(x+1)(x-1)}$$

$$\Rightarrow A + B = 0, \quad B - A = 1$$

$$\Rightarrow A = \frac{-1}{2}, \quad B = \frac{1}{2}$$

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \left[\frac{1}{x-1} - \frac{1}{x+1}\right] dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1| + C \qquad \square$$

1. (a)

$$\int \frac{1}{\sqrt{x-1} + \sqrt{x+1}}\,dx = \int \frac{\sqrt{x-1} - \sqrt{x+1}}{(x-1) - (x+1)}\,dx$$

$$= \frac{1}{2}\int \left[\sqrt{x+1} - \sqrt{x-1}\,\right] dx$$

$$= \frac{1}{3}(x+1)^{3/2} - \frac{1}{3}(x-1)^{3/2} + C \qquad \square$$

1. (b) *I am not typesetting polynomial long division; but you must write it out.*

$$\int \frac{8x^2 + 6x + 4}{x+1}\,dx = \int \left[8x - 2 + \frac{6}{x+1}\right] dx$$

$$= 4x^2 - 2x + 6\log|x+1| + C \qquad \square$$

1\. (c)

$$\int \tan^2(x)\,dx = \int \left[\sec^2(x) - 1\right]\,dx = \tan(x) - x + C \qquad \square$$

1\. (d)

$$\int \frac{1}{1+\sin(x)}\,dx = \int \frac{1-\sin(x)}{1-\sin^2(x)}\,dx = \int \frac{1-\sin(x)}{\cos^2(x)}\,dx$$

$$= \int \left[\sec^2(x) - \sec(x)\tan(x)\right]\,dx = \tan(x) - \sec(x) + C \qquad \square$$

2\. (a)

$$\int \frac{e^x}{e^{2x}+2e^x+1}\,dx = \int \frac{e^x}{(e^x+1)^2}\,dx \quad \begin{bmatrix}\text{Let } u = e^x\\ du = e^x\,dx\end{bmatrix}$$

$$= \int \frac{1}{(u+1)^2}\,du = \frac{-1}{u+1} + C = \frac{-1}{e^x+1} + C \qquad \square$$

2\. (b)

$$\int \frac{x}{\sqrt{1-x^4}}\,dx \quad \begin{bmatrix}\text{Let } u = x^2\\ du = 2\,dx\end{bmatrix} = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(x^2) + C \qquad \square$$

2\. (c)

$$\int \log\left(\cos(x)\right)\tan(x)\,dx \quad \begin{bmatrix}\text{Let } u = \log(\cos(x))\\ du = -\tan(x)\,dx\end{bmatrix}$$

$$= -\int u\,du = \frac{-1}{2}u^2 + C = \frac{-1}{2}\log^2(\cos(x)) + C \qquad \square$$

3\. (a)

$$\int \sqrt{x}\log(x)\,dx \quad \begin{bmatrix}u = \log(x) & dv = \sqrt{x}\,dx\\ du = \frac{1}{x}\,dx & v = \frac{2}{3}x^{3/2}\end{bmatrix}$$

$$= \frac{2}{3}x^{3/2}\log(x) - \frac{2}{3}\int \sqrt{x}\,dx$$

$$= \frac{2}{3}x^{3/2}\log(x) - \frac{4}{9}x^{3/2} + C \qquad \square$$

3\. (b)

$$\int x^2 e^x\,dx \quad \begin{bmatrix}u = x^2 & dv = e^x\,dx\\ du = 2x\,dx & v = e^x\end{bmatrix}$$

$$= x^2e^x - 2\int xe^x\,dx \quad \begin{bmatrix}u = x & dv = e^x\,dx\\ du = dx & v = e^x\end{bmatrix}$$

$$= x^2e^x - 2xe^x + 2\int e^x\,dx$$

$$= (x^2 - 2x + 2)e^x + C \qquad \square$$

3. (c)
$$\int \log^3(x)\,dx \quad \begin{bmatrix} u = \log^3(x) & dv = dx \\ du = \dfrac{3\log^2(x)}{x}\,dx & v = x \end{bmatrix}$$
$$= x\log^3(x) - 3\int \log^2(x)\,dx \quad \begin{bmatrix} u = \log^2(x) & dv = dx \\ du = \dfrac{2\log(x)}{x}\,dx & v = x \end{bmatrix}$$
$$= x\log^3(x) - 3x\log^2(x) + 6\int \log(x)\,dx \quad \begin{bmatrix} u = \log(x) & dv = dx \\ du = \dfrac{1}{x}\,dx & v = x \end{bmatrix}$$
$$= x\log^3(x) - 3x\log^2(x) + 6x\log(x) - 6\int dx$$
$$= x\log^3(x) - 3x\log^2(x) + 6x\log(x) - 6x + C \qquad \square$$

3. (d)
$$\int \sec^3(x)\,dx \quad \begin{bmatrix} u = \sec(x) & dv = \sec^2(x)\,dx \\ du = \sec(x)\tan(x)\,dx & v = \tan(x)\,dx \end{bmatrix}$$
$$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$$
$$= \sec(x)\tan(x) - \int \sec(x)\left(\sec^2(x) - 1\right)\,dx$$
$$= \sec(x)\tan(x) + \int \sec(x)\,dx - \int \sec^3(x)\,dx$$
$$= \sec(x)\tan(x) + \log|\sec(x) + \tan(x)| - \int \sec^3(x)\,dx$$
$$\Rightarrow 2\int \sec^3(x) = \sec(x)\tan(x) + \log|\sec(x) + \tan(x)|$$
$$\Rightarrow \int \sec^3(x) = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\log|\sec(x) + \tan(x)| + C \qquad \square$$

4. (a)
$$\int \frac{x+4}{x^2+1}\,dx = 4\int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$
$$= 4\arctan(x) + \frac{1}{2}\int \frac{2x}{x^2+1} \quad \begin{bmatrix} \text{Let } u = x^2 + 1 \\ du = 2x\,dx \end{bmatrix}$$
$$= 4\arctan(x) + \frac{1}{2}\int \frac{1}{u}\,du$$
$$= 4\arctan(x) + \frac{1}{2}\log|u| + C$$
$$= 4\arctan(x) + \frac{1}{2}\log|x^2 + 1| + C \qquad \square$$

4. (b)

$$\frac{2x^2+7x-1}{x^3+x^2-x-1} = \frac{2x^2+7x-1}{(x-1)(x+1)^2}$$
$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
$$= \frac{A(x^2+2x+1)+B(x^2-1)+C(x-1)}{(x-1)(x+1)^2}$$
$$= \frac{(A+B)x^2+(2A+C)x+(A-B-C)}{(x-1)(x+1)^2}$$
$$\Rightarrow A+B=2, \quad 2A+C=7, \quad A-B-C=-1$$
$$\Rightarrow A=2, \quad B=0, \quad C=3$$

$$\int \frac{2x^2+7x-1}{x^3+x^2-x-1}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{3}{(x+1)^2}\,dx$$
$$= 2\log|x-1| - \frac{3}{x+1} + C$$

□

4. (c)

$$\frac{x^3+x+2}{x^4+2x^2+1} = \frac{x^3+x+2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$
$$= \frac{(Ax+B)(x^2+1)+Cx+D}{(x^2+1)^2}$$
$$= \frac{Ax^3+Bx^2+(A+C)x+(B+D)}{(x^2+1)^2}$$
$$\Rightarrow A=1, \quad B=0, \quad A+C=1, \quad B+D=2$$
$$\Rightarrow A=1, \quad B=0, \quad C=0, \quad D=2$$

$$\int \frac{x^3+x+2}{x^4+2x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{(x^2+1)^2}\,dx = (*)$$

$$\int \frac{x}{x^2+1}\,dx \begin{bmatrix}\text{Let } u = x^2+1 \\ du = 2x\,dx\end{bmatrix} = \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\log|u| + C = \frac{1}{2}\log|x^2+1| + C$$

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{(x^2+1)-x^2}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x^2+1}\,dx - \int \frac{x^2}{(x^2+1)^2}\,dx \quad \begin{bmatrix} u = x & dv = \frac{x}{(x^2+1)^2}\,dx \\ du = dx & v = \frac{1}{2}\frac{-1}{(x^2+1)} \end{bmatrix}$$

$$= \arctan(x) + \frac{1}{2}\frac{x}{x^2+1} - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{x}{x^2+1} + C$$

$$(*) = \frac{1}{2}\log|x^2+1| + \arctan(x) + \frac{x}{x^2+1} + C \qquad \square$$

(This was, after all, the last problem on the assignment.)