A Counterintuitive Phenomenon in Rigid Body Motion

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June 2024

1 Introduction

Consider the following system of ODEs for $(X(t), Y(t), Z(t))$:

$$\begin{aligned} \frac{dX}{dt} &= -YZ - \nu X, \\ \frac{dY}{dt} &= 1 + 2XZ - \nu Y, \\ \frac{dZ}{dt} &= -XY - \nu Z. \end{aligned} \tag{1}$$

This system arises as a very low-dimensional approximation for fluid motion, being forced at the "intermediate scale" Y. There is a steady state, $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$. The first aim is to show that it is stable for a large ν and that it goes unstable for a small ν. It is also a goal to understand what "secondary flows" (different family of steady states) arise and exchange stability with that one.

2 Energy

Before discussing stability, we first introduce a concept.

Definition 2.1 (Energy). *$X^2 + Y^2 + Z^2$ is called the energy of the system.*

Consider the "zero force, zero dissipation" version of the system we are to study:

$$\begin{aligned} \frac{dX}{dt} &= -YZ, \\ \frac{dY}{dt} &= 2XZ, \\ \frac{dZ}{dt} &= -XY. \end{aligned} \tag{2}$$

Theorem 2.1. *The energy of the system (2), $X^2 + Y^2 + Z^2$, is preserved over time.*

Proof. $\frac{d(X^2+Y^2+Z^2)}{dt} = 2(X\frac{dX}{dt} + Y\frac{dY}{dt} + Z\frac{dZ}{dt}) = 2[X(-YZ) + Y(2XZ) + Z(-XY)] = 2(-XYZ + 2XYZ - XYZ) = 0$. □

The energy gives us inspiration to prove the stability in the following, since it can be defined as a Lyapunov function which will be introduced. Although the energy of the system (1) is not preserved in time, it is still very useful.

3 Stability for $\nu \geq 1$

Now we introduce the definition of various stability.

Definition 3.1 (Stability)**.** *A solution ψ of a system*

$$\frac{dx}{dt} = F(t, x)$$

which is defined for $t \geq 0$ is said to be stable if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that any solution ϕ of the system satisfying

$$|\phi(0) - \psi(0)| < \delta$$

satisfies

$$|\phi(t) - \psi(t)| < \epsilon \qquad (t \geq 0).$$

Note that this requires that solutions starting near $\psi(0)$ exist for all $t \geq 0$.

Definition 3.2 (Asymptotic stability)**.** *A solution ψ of a system*

$$\frac{dx}{dt} = F(t, x)$$

which is defined for $t \geq 0$ is said to be asymptotically stable if, ψ is stable, and there exists a $\delta > 0$ such that any solution ϕ of the system satisfying

$$|\phi(0) - \psi(0)| < \delta$$

satisfies

$$|\phi(t) - \psi(t)| \to 0 \qquad (t \to \infty).$$

Definition 3.3 (Global asymptotic stability)**.** *A solution ψ of a system*

$$\frac{dx}{dt} = F(t, x)$$

which is defined for $t \geq 0$ is said to be globally asymptotically stable if, ψ is stable, and for any $\phi(0) \in \mathbb{R}$,

$$|\phi(t) - \psi(t)| \to 0 \qquad (t \to \infty).$$

The steady state $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$ of (1) shows stability for $\nu \geq 1$.

Theorem 3.1. *The steady state of (1), $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$, is asymptotically stable for $\nu > 1$.*

We use the method of linearizing the system of ODEs to prove this theorem.

Lemma 3.1.1. *Let*

$$\frac{dx}{dt} = Ax + f(t, x)$$

where A is a real constant matrix with the characteristic roots all having negative real parts. Let f be real continuous for small $|x|$ and $t \geq 0$, and

$$f(t, x) = o(|x|) \qquad (|x| \to 0)$$

uniformly in t, $t \geq 0$. Then the identically zero solution is asymptotically stable.

Please refer to [1] for the proof of this lemma.

Proof of theorem 3.1. To use lemma 3.1.1, we need to transfer the steady state of (1), $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$, to be the identically zero solution at first, so we need to transform variables.

Let $x = X$, $y = Y - \frac{1}{\nu}$, $z = Z$, then $X = x$, $Y = y + \frac{1}{\nu}$, $Z = z$.

Therefore,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dX}{dt} = -YZ - \nu X = -(y + \frac{1}{\nu})z - \nu x = -yz - \frac{1}{\nu}z - \nu x, \\ \frac{dy}{dt} &= \frac{dY}{dt} = 1 + 2XZ - \nu Y = 1 + 2xz - \nu(y + \frac{1}{\nu}) = 2xz - \nu y, \\ \frac{dz}{dt} &= \frac{dZ}{dt} = -XY - \nu Z = -x(y + \frac{1}{\nu}) - \nu z = -xy - \frac{1}{\nu}x - \nu z. \end{aligned} \tag{3}$$

Namely,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\nu & & -\frac{1}{\nu} \\ & -\nu & \\ -\frac{1}{\nu} & & -\nu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -yz \\ 2xz \\ -xy \end{pmatrix}. \tag{4}$$

Let $A = \begin{pmatrix} -\nu & & -\frac{1}{\nu} \\ & -\nu & \\ -\frac{1}{\nu} & & -\nu \end{pmatrix}$, then

$$|\lambda I - A| = \begin{vmatrix} \lambda + \nu & & \frac{1}{\nu} \\ & \lambda + \nu & \\ \frac{1}{\nu} & & \lambda + \nu \end{vmatrix} = (\lambda + \nu)^3 - \frac{1}{\nu^2}(\lambda + \nu) = [(\lambda + \nu)^2 - \frac{1}{\nu^2}](\lambda + \nu).$$

Let $g(\lambda) = (\lambda + \nu)^2 - \frac{1}{\nu^2}$, then $g(0) = \nu^2 - \frac{1}{\nu^2} > 0$ for $\nu > 1$. Therefore, $g(\lambda)$ has two negative roots.

The characteristic roots of A are the two negative roots of $g(\lambda)$ and $-\nu$, so they are all negative.

Let $f(t,x,y,z) = \begin{pmatrix} -yz \\ 2xz \\ -xy \end{pmatrix}$, then $\lim\limits_{|(x,y,z)|\to 0} \frac{1}{|(x,y,z)|} f(t,x,y,z)$

$$= \lim_{(x,y,z)\to 0} \frac{1}{\sqrt{x^2+y^2+z^2}} \begin{pmatrix} -yz \\ 2xz \\ -xy \end{pmatrix} = \lim_{(x,y,z)\to 0} \begin{pmatrix} -\frac{sgn(yz)}{\sqrt{\frac{x^2}{y^2z^2}+\frac{1}{z^2}+\frac{1}{y^2}}} \\ \frac{2sgn(xz)}{\sqrt{\frac{1}{z^2}+\frac{y^2}{x^2z^2}+\frac{1}{x^2}}} \\ -\frac{sgn(xy)}{\sqrt{\frac{1}{y^2}+\frac{1}{x^2}+\frac{z^2}{x^2y^2}}} \end{pmatrix} = 0.$$

Therefore,

$$f(t,x,y,z) = o(|(x,y,z)|) \qquad (|(x,y,z)| \to 0)$$

uniformly in t, $t \geq 0$.

By lemma 3.1.1, $x = 0$, $y = 0$, $z = 0$ is asymptotically stable, so $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$ is asymptotically stable. □

Theorem 3.2. *The steady state of (1), $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$, is stable for $\nu = 1$.*

We introduce a concept called Lyapunov function to prove this theorem.

Definition 3.4 (Lyapunov function)**.** *Suppose $\frac{dx}{dt} = f(x)$, $f(0) = 0$. $V : \mathbb{R}^n \to \mathbb{R}$ is called a Lyapunov function if there exists an open set $R \ni 0$ such that*

1) $V(x)$ is continuous in R;

2) For any $x \in R\backslash\{0\}$, $V(x) > 0$;

3) $V(0) = 0$;

4) For any $x \in R$, $V'(x) = \nabla V(x) \cdot f(x) \leq 0$.

Lemma 3.2.1. *If there exists a Lyapunov function, then the origin is stable.*

Proof. Let $S_\epsilon = \{x \in \mathbb{R} \mid |x| = \epsilon\}$ for small $\epsilon > 0$ such that $S_\epsilon \in R$.

V attains minimum on S_ϵ. Call it $m > 0$.

Since V is continuous in R, there exists $\delta > 0$ such that if $|x| < \delta$, then $V(x) < m$.

Suppose $|x(0)| < \delta$, then $V(x(0)) < m$. Then for any $t \geq 0$, $V(x(t)) \leq V(x(0)) < m$.

Assume $|x(t)| = \epsilon$ for some t, then $V(x(t)) \geq m$. Contradiction!

Hence $|x(t)| < \epsilon$, so the origin is stable. □

Proof of theorem 3.2. We implement the same transformation of variables.

Let $x = X$, $y = Y - \frac{1}{\nu}$, $z = Z$, then we will get (3).

Let $V(x,y,z) = x^2 + y^2 + z^2$, then

1) $V(x,y,z)$ is continuous in $\mathbb{R}^3$;

2) For any $(x,y,z) \in \mathbb{R}^3\backslash\{0\}$, $V(x,y,z) > 0$;

3) $V(0,0,0) = 0$;

4) For any $x \in \mathbb{R}^3$, $V'(x) = 2(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}) = 2[-(x+z)^2 - y^2] \leq 0$.

Therefore, $V(x,y,z)$ is a Lyapunov function. By lemma 3.2.1, $x=0$, $y=0$, $z=0$ is stable, so $X=0$, $Y=\frac{1}{\nu}$, $Z=0$ is stable. □

Now we have proved the stability of the steady state of (1) $X=0$, $Y=\frac{1}{\nu}$, $Z=0$ for $\nu \geq 1$. However, there is a stronger conclusion.

Theorem 3.3. *The steady state of (1), $X=0$, $Y=\frac{1}{\nu}$, $Z=0$, is globally asymptotically stable for $\nu \geq 1$.*

Lemma 3.3.1 (LaSalle's invariance principle)**.** *Suppose a system is represented as $\frac{dx}{dt}=f(x)$ where x is the vector of variables, with $f(0)=0$. Let $\mathcal{I}$ be the union of complete trajectories contained entirely in the set $\{x \mid V'(x)=0\}$. If a C^1 function $V(x)$ can be found such that*

1) $V'(x)$ is negative semidefinite, i.e. $V'(x) \leq 0$ for all x;

2) $V(x)$ is positive definite, i.e.

i) $V(x)>0$ for all $x \neq 0$;

ii) $V(0)=0$;

3) $V(x)$ is radially unbounded, i.e. $V(x) \to \infty$, as $||x|| \to \infty$;

and if $\mathcal{I}$ contains no trajectory of the system except the trivial trajectory $x(t)=0$ for $t \geq 0$, then the origin is globally asymptotically stable.

Please refer to [2] for the proof of this lemma.

Proof of theorem 3.3. Let $x=X$, $y=Y-\frac{1}{\nu}$, $z=Z$, then we will get (3).

Let $V(x,y,z)=x^2+y^2+z^2$, then

1) $V'(x,y,z)=2(x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt})=2(-\frac{2}{\nu}xz-\nu(x^2+y^2+z^2)) \leq 2(-\frac{2}{\nu}xz-2\nu|xz|-\nu y^2)$.

If $xz \geq 0$, then $-\frac{2}{\nu}xz-2\nu|xz|-\nu y^2=-\frac{2}{\nu}xz-2\nu xz-\nu y^2 \leq 0$;

If $xz<0$, then $-\frac{2}{\nu}xz-2\nu|xz|-\nu y^2=-\frac{2}{\nu}xz+2\nu xz-\nu z^2=2(\nu-\frac{1}{\nu})xz-\nu y^2 \leq 2(1-1)xz \leq 0$.

Hence $V'(x,y,z) \leq 0$ for all (x,y,z);

2) $V(x,y,z)>0$ for all $(x,y,z) \neq 0$, and $V(0,0,0)=0$;

3) $V(x,y,z) \to \infty$, as $||(x,y,z)|| \to \infty$.

Let $V'(x,y,z)=0$, then $V'(x,y,z)=2(-\frac{2}{\nu}xz-\nu(x^2+y^2+z^2))=2(-\frac{2}{\nu}xz-2\nu|xz|-\nu y^2)=0$, so $|x|=|z|$.

If $xz \geq 0$, then $x=y=z=0$;

If $xz<0$ and $\nu>1$, then $x=y=z=0$;

If $xz<0$ and $\nu=1$, then $x=-z$, $y=0$. By (3), $\frac{dy}{dt}=2xz-\nu y=2xz=0$, so $x=0$ or $z=0$. Whether $x=0$ or $z=0$, $x=z=0$.

Hence $\{(x,y,z) \mid V'(x,y,z)=0\}=\{0\}$.

By lemma 3.3.1, $x=0$, $y=0$, $z=0$ is globally asymptotically stable, so $X=0$, $Y=\frac{1}{\nu}$, $Z=0$ is globally asymptotically stable. □

4 instability for $0<\nu<1$

The steady state $X=0$, $Y=\frac{1}{\nu}$, $Z=0$ of (1) shows instability for $0<\nu<1$.

Theorem 4.1. *The steady state of (1), $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$, is unstable for $0 < \nu < 1$.*

Similarly to theorem 3.1, We use the method of linearizing the system of ODEs to prove this theorem.

Lemma 4.1.1. *Let*

$$\frac{dx}{dt} = Ax + f(t, x)$$

where A is a real constant matrix with at least one characteristic root having positive real part. Let f be real continuous for small $|x|$ and $t \geq 0$, and

$$f(t, x) = o(|x|) \qquad (|x| \to 0)$$

uniformly in t, $t \geq 0$. Then the identically zero solution is unstable.

Please refer to [1] for the proof of this lemma.

Proof of theorem 4.1. Let $x = X$, $y = Y - \frac{1}{\nu}$, $z = Z$, then we will get (4).

Let $A = \begin{pmatrix} -\nu & & -\frac{1}{\nu} \\ & -\nu & \\ -\frac{1}{\nu} & & -\nu \end{pmatrix}$, then

$$|\lambda I - A| = [(\lambda + \nu)^2 - \frac{1}{\nu^2}](\lambda + \nu).$$

Let $g(\lambda) = (\lambda + \nu)^2 - \frac{1}{\nu^2}$, then $g(0) = \nu^2 - \frac{1}{\nu^2} < 0$ for $0 < \nu < 1$. Therefore, $g(x)$ has a positive root.

The characteristic roots of A are the two roots of $g(x)$ and $-\nu$, so a characteristic root of A is positive.

Let $f(t, x, y, z) = \begin{pmatrix} -yz \\ 2xz \\ -xy \end{pmatrix}$, then similarly to theorem 3.1,

$$f(t, x, y, z) = o(|(x, y, z)|) \qquad (|(x, y, z)| \to 0)$$

uniformly in t, $t \geq 0$.

By lemma 3.1.1, $x = 0$, $y = 0$, $z = 0$ is unstable, so $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$ is unstable. □

5 The Other Two Steady States for $0 < \nu < 1$

For (1), if we let $\frac{dX}{dt} = \frac{dY}{dt} = \frac{dZ}{dt} = 0$, then we will find another two steady states for $0 < \nu < 1$: $X = \sqrt{\frac{1-\nu^2}{2}}$, $Y = \nu$, $Z = -\sqrt{\frac{1-\nu^2}{2}}$ and $X = -\sqrt{\frac{1-\nu^2}{2}}$, $Y = \nu$, $Z = \sqrt{\frac{1-\nu^2}{2}}$. Actually, they show stability.

Theorem 5.1. *The steady states of (1), $X = \sqrt{\frac{1-\nu^2}{2}}$, $Y = \nu$, $Z = -\sqrt{\frac{1-\nu^2}{2}}$ and $X = -\sqrt{\frac{1-\nu^2}{2}}$, $Y = \nu$, $Z = \sqrt{\frac{1-\nu^2}{2}}$ are asymptotically stable for* $0 < \nu < 1$.

Similarly to theorem 3.1, We use the method of linearizing the system of ODEs to prove this theorem.

Proof. Let $x = X - \sqrt{\frac{1-\nu^2}{2}}$, $y = Y - \nu$, $z = Z + \sqrt{\frac{1-\nu^2}{2}}$, then $X = x + \sqrt{\frac{1-\nu^2}{2}}$, $Y = y + \nu$, $Z = z - \sqrt{\frac{1-\nu^2}{2}}$.

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dX}{dt} = -YZ - \nu X = -\nu x + \sqrt{\frac{1-\nu^2}{2}}y - \nu z - yz,\\
\frac{dy}{dt} &= \frac{dY}{dt} = 1 + 2XZ - \nu Y = -\sqrt{2(1-\nu^2)}x - \nu y + \sqrt{2(1-\nu^2)}z + 2xz,\\
\frac{dz}{dt} &= \frac{dZ}{dt} = -XY - \nu Z = -\nu x - \sqrt{\frac{1-\nu^2}{2}}y - \nu z - xy.
\end{aligned}$$

Namely,

$$\frac{d}{dt}\begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} -\nu & \sqrt{\frac{1-\nu^2}{2}} & -\nu\\ -\sqrt{2(1-\nu^2)} & -\nu & \sqrt{2(1-\nu^2)}\\ -\nu & -\sqrt{\frac{1-\nu^2}{2}} & -\nu\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix} + \begin{pmatrix} -yz\\ 2xz\\ -xy\end{pmatrix}.$$

Let $A = \begin{pmatrix} -\nu & \sqrt{\frac{1-\nu^2}{2}} & -\nu\\ -\sqrt{2(1-\nu^2)} & -\nu & \sqrt{2(1-\nu^2)}\\ -\nu & -\sqrt{\frac{1-\nu^2}{2}} & -\nu\end{pmatrix}$, then

$$|\lambda I - A| = \begin{vmatrix} \lambda+\nu & -\sqrt{\frac{1-\nu^2}{2}} & \nu\\ \sqrt{2(1-\nu^2)} & \lambda+\nu & -\sqrt{2(1-\nu^2)}\\ \nu & \sqrt{\frac{1-\nu^2}{2}} & \lambda+\nu\end{vmatrix} = (\lambda+2\nu)(\lambda^2+\nu\lambda+2-2\nu^2).$$

Let $g(\lambda) = \lambda^2 + \nu\lambda + 2 - 2\nu^2 = (x + \frac{\nu}{2})^2 + 2 - \frac{9}{4}\nu^2$, then $g(0) = 2 - 2\nu^2 > 0$.
If $2 - \frac{9}{4}\nu^2 \leq 0$, then $g(x)$ has two negative roots;
If $2 - \frac{9}{4}\nu^2 > 0$, then $g(x)$ has two imaginary roots with negative real part $-\frac{1}{2}\nu$.
The characteristic roots of A are the two roots of $g(x)$ and -2ν, so their real parts are all negative.

Let $f(t, x, y, z) = \begin{pmatrix} -yz\\ 2xz\\ -xy\end{pmatrix}$, then similarly to theorem 3.1,

$$f(t, x, y, z) = o(|(x, y, z)|) \qquad (|(x, y, z)| \to 0)$$

uniformly in t, $t \geq 0$.

By lemma 3.1.1, $x = 0$, $y = 0$, $z = 0$ is asymptotically stable, so $X = \sqrt{\frac{1-\nu^2}{2}}$, $Y = \nu$, $Z = -\sqrt{\frac{1-\nu^2}{2}}$ is asymptotically stable.

Similarly, $X = -\sqrt{\frac{1-\nu^2}{2}}$, $Y = \nu$, $Z = \sqrt{\frac{1-\nu^2}{2}}$ is also asymptotically stable.

□

We can use the plane $Z = X$ to divide the whole space into two half-spaces and a plane, then we will get some more stronger results.

Theorem 5.2. *The steady state of (1), $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$, is globally asymptotically stable on the plane $Z = X$ for $0 < \nu < 1$. Namely, take any point on the plane $Z = X$ as the initial condition, the trajectory will tend to $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$, as $t \to \infty$.*

Similarly to theorem 3.3, we use lemma 3.3.1 LaSalle's invariance principle to prove this theorem.

Proof. A normal vector of the plane $Z = X$ is $(1, 0, -1)$. The vector space of the the original system (1) on the plane $Z = X$ is $(\frac{dX}{dt}, \frac{dY}{dt}, \frac{dZ}{dt}) = (-YZ - \nu X, 1 + 2XZ - \nu Y, -XY - \nu Z) = (-XY - \nu X, 1 + 2X^2 - \nu Y, -XY - \nu X)$.

$(1, 0, -1) \cdot (\frac{dX}{dt}, \frac{dY}{dt}, \frac{dZ}{dt}) = (-XY - \nu X) - (-XY - \nu X) = 0$, so $(1, 0, -1) \perp (\frac{dX}{dt}, \frac{dY}{dt}, \frac{dZ}{dt})$.

Hence the vector space of the the system (1) on the plane $Z = X$ is parallel to the plane $Z = X$ itself, which means that any trajectory of the system (1) whose initial value is a point on the plane $Z = X$ keeps on that plane for any t.

We project the system (1) on the plane $Z = X$ onto the XY-plane, then it becomes a 2-dimensional system:

$$\begin{aligned} \frac{dX}{dt} &= -XY - \nu X, \\ \frac{dY}{dt} &= 1 + 2X^2 - \nu Y. \end{aligned}$$

Let $x = X$, $y = Y - \frac{1}{\nu}$, then $X = x$, $Y = y + \frac{1}{\nu}$.
Therefore,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dX}{dt} = -XY - \nu X = -x(y + \frac{1}{\nu}) - \nu x = -xy - (\nu + \frac{1}{\nu})x, \\ \frac{dy}{dt} &= \frac{dY}{dt} = 1 + 2X^2 - \nu Y = 1 + 2x^2 - \nu(y + \frac{1}{\nu}) = 2x^2 - \nu y. \end{aligned} \tag{5}$$

Let $V(x, y) = 2x^2 + y^2$, then
1) $V'(x, y) = 2(2x\frac{dx}{dt} + y\frac{dy}{dt}) = 2(-2(\nu + \frac{1}{\nu})x^2 - \nu y^2) \leq 0$ for all (x, y);
2) $V(x, y) > 0$ for all $(x, y) \neq 0$, and $V(0, 0) = 0$;
3) $V(x, y) \to \infty$, as $||(x, y)|| \to \infty$.
Moreover, $\{(x, y)|V'(x, y) = 0\} = \{0\}$.

By lemma 3.3.1, $x = 0$, $y = 0$ is globally asymptotically stable, so $X = 0$, $Y = \frac{1}{\nu}$ is globally asymptotically stable for the system (5).

Hence $X = 0$, $Y = \frac{1}{\nu}$, $Z = 0$, is globally asymptotically stable on the plane $Z = X$ for the system (1). □

I suppose that $X = \sqrt{\frac{1-\nu^2}{2}}$, $Y = \nu$, $Z = -\sqrt{\frac{1-\nu^2}{2}}$ is globally asymptotically stable in the half-space $Z < X$ and $X = -\sqrt{\frac{1-\nu^2}{2}}$, $Y = \nu$, $Z = \sqrt{\frac{1-\nu^2}{2}}$ is globally asymptotically stable in the half-space $Z < X$, and the following lemma may be used to prove these two conjectures.

Lemma 5.2.1. *(Local invariant set theorem) Consider an autonomous system of the form*

$$\frac{dx}{dt} = f(x),$$

with f continuous, and let $V(x)$ be a scalar function with continuous first partial derivatives. Assume that 1) for some $l > 0$, the region Ω_l defined by $V(x) < l$ is bounded;

2) $V'(x) \leq 0$ for all x in Ω_l.

Let R be the set of all points within Ω_l where $V'(x) = 0$, and M be the largest invariant set in R. Then, every solution $x(t)$ originating in Ω_l tends to M as $t \to \infty$.

Please refer to [3] for the proof of this lemma.

6 Generalization

Let $I_1, I_2, I_3 > 0$ be the given moments of inertia. The equations of motion for the rigid body with angular velocities $(\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ about its moment of inertia axes are

$$\begin{aligned} I_1 \frac{d\Omega_1}{dt} &= (I_2 - I_3)\Omega_2\Omega_3, \\ I_2 \frac{d\Omega_2}{dt} &= (I_3 - I_1)\Omega_1\Omega_3, \\ I_3 \frac{d\Omega_3}{dt} &= (I_1 - I_2)\Omega_1\Omega_2. \end{aligned}$$

Without loss of generality, we assume $I_1 \leq I_2 \leq I_3$. We study a slightly more general class of systems, which include the addition of linear friction and body forcing.

$$\begin{aligned} I_1 \frac{d\Omega_1}{dt} &= a\Omega_2\Omega_3 - \nu\Omega_1, \\ I_2 \frac{d\Omega_2}{dt} &= b\Omega_1\Omega_3 - \nu\Omega_2 + 1, \\ I_3 \frac{d\Omega_3}{dt} &= c\Omega_1\Omega_2 - \nu\Omega_3, \end{aligned} \tag{6}$$

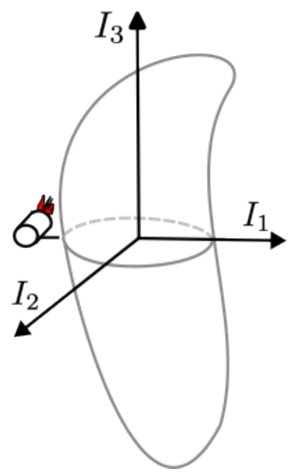

Figure 1: Rigid body with booster rocket forcing rotation about I_3 axis

where $a+b+c=0$ with $a,c<0$ and $b>0$.
For $\nu \geq \sqrt[4]{ac}$, there is exactly one root

$$(\Omega_1^*, \Omega_2^*, \Omega_3^*) = (0, \frac{1}{\nu}, 0).$$

For $0 < \nu < \sqrt[4]{ac}$, there are exactly three real roots

$$\begin{aligned}
(\Omega_1^*, \Omega_2^*, \Omega_3^*) &= (0, \frac{1}{\nu}, 0), \\
(\Omega_1^+, \Omega_2^+, \Omega_3^+) &= (\sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}, \frac{\nu}{\sqrt{ac}}, -\sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}}), \\
(\Omega_1^-, \Omega_2^-, \Omega_3^-) &= (-\sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}, \frac{\nu}{\sqrt{ac}}, \sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}}).
\end{aligned}$$

Theorem 6.1. *The steady state of (6), $(\Omega_1^*, \Omega_2^*, \Omega_3^*) = (0, \frac{1}{\nu}, 0)$, is globally asymptotically stable for $\nu \geq \sqrt[4]{ac}$.*

Proof. Let $x = \Omega_1$, $y = \Omega_2 - \frac{1}{\nu}$, $z = \Omega_3$, then $\Omega_1 = x$, $\Omega_2 = y + \frac{1}{\nu}$, $\Omega_3 = z$.
Therefore,

$$\begin{aligned}
I_1 \frac{dx}{dt} &= I_1 \frac{d\Omega_1}{dt} = a\Omega_2\Omega_3 - \nu\Omega_1 = -\nu x + \frac{a}{\nu} z + ayz, \\
I_2 \frac{dy}{dt} &= I_2 \frac{d\Omega_2}{dt} = b\Omega_1\Omega_3 - \nu\Omega_2 + 1 = -\nu y + bxz, \qquad (7) \\
I_3 \frac{dz}{dt} &= I_3 \frac{d\Omega_3}{dt} = c\Omega_1\Omega_2 - \nu\Omega_3 = \frac{c}{\nu} x - \nu z + cxy.
\end{aligned}$$

Let $V(x,y,z) = -\frac{I_1}{a}x^2 + 2\frac{I_2}{b}y^2 - \frac{I_3}{c}z^2$, then

1) $V'(x,y,z) = 2(-\frac{I_1}{a}x\frac{dx}{dt} + 2\frac{I_2}{b}y\frac{dy}{dt} - \frac{I_3}{c}z\frac{dz}{dt}) = 2(\frac{\nu}{a}x^2 - \frac{2\nu}{b}y^2 + \frac{\nu}{c}z^2 - \frac{2}{\nu}xz) \leq 2(-\frac{2\nu}{\sqrt{ac}}|xz| - \frac{2}{\nu}xz - \frac{2\nu}{b}y^2)$.

If $xz \geq 0$, then $-\frac{2\nu}{\sqrt{ac}}|xz| - \frac{2}{\nu}xz - \frac{2\nu}{b}y^2 = -\frac{2\nu}{\sqrt{ac}}xz - \frac{2}{\nu}xz - \frac{2\nu}{b}y^2 \leq 0$;

If $xz < 0$, then $-\frac{2\nu}{\sqrt{ac}}|xz| - \frac{2}{\nu}xz - \frac{2\nu}{b}y^2 = \frac{2\nu}{\sqrt{ac}}xz - \frac{2}{\nu}xz - \frac{2\nu}{b}y^2 = 2(\frac{\nu}{\sqrt{ac}} - \frac{1}{\nu})xz - \frac{2\nu}{b}y^2 \leq 2(\frac{\sqrt[4]{ac}}{\sqrt{ac}} - \frac{1}{\sqrt[4]{ac}})xz = 0$.

Hence $V'(x,y,z) \leq 0$ for all (x,y,z);

2) $V(x,y,z) > 0$ for all $(x,y,z) \neq 0$, and $V(0,0,0) = 0$;

3) $V(x,y,z) \to \infty$, as $||(x,y,z)|| \to \infty$.

Let $V'(x,y,z) = 0$, then $V'(x,y,z) = 2(\frac{\nu}{a}x^2 - \frac{2\nu}{b}y^2 + \frac{\nu}{c}z^2 - \frac{2}{\nu}xz) = 2(-\frac{2\nu}{\sqrt{ac}}|xz| - \frac{2}{\nu}xz - \frac{2\nu}{b}y^2) = 0$, so $\sqrt{-c}|x| = \sqrt{-a}|z|$.

If $xz \geq 0$, then $x = y = z = 0$; If $xz < 0$ and $\nu > \sqrt[4]{ac}$, then $x = y = z = 0$; If $xz < 0$ and $\nu = \sqrt[4]{ac}$, then $\sqrt{-c}x = -\sqrt{-a}z$, $y = 0$. By (7), $I_2\frac{dy}{dt} = -\nu y + bxz = bxz = 0$, so $x = 0$ or $z = 0$. Whether $x = 0$ or $z = 0$, $x = z = 0$.

Hence $\{(x,y,z) \mid V'(x,y,z) = 0\} = \{0\}$.

By lemma 3.3.1, $x = 0$, $y = 0$, $z = 0$ is globally asymptotically stable, so $(\Omega_1^*, \Omega_2^*, \Omega_3^*) = (0, \frac{1}{\nu}, 0)$ is globally asymptotically stable. □

Theorem 6.2. *The steady state of (6), $(\Omega_1^*, \Omega_2^*, \Omega_3^*) = (0, \frac{1}{\nu}, 0)$, is unstable for $0 < \nu < \sqrt[4]{ac}$.*

Proof. Let $x = \Omega_1$, $y = \Omega_2 - \frac{1}{\nu}$, $z = \Omega_3$, then

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{\nu}{I_1} & & \frac{a}{\nu I_1} \\ & -\frac{\nu}{I_2} & \\ \frac{c}{\nu I_3} & & -\frac{\nu}{I_3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{ayz}{I_1} \\ \frac{bxz}{I_2} \\ \frac{cxy}{I_3} \end{pmatrix}.$$

Let $A = \begin{pmatrix} -\frac{\nu}{I_1} & & \frac{a}{\nu I_1} \\ & -\frac{\nu}{I_2} & \\ \frac{c}{\nu I_3} & & -\frac{\nu}{I_3} \end{pmatrix}$, then

$$|\lambda I - A| = [(\lambda + \frac{\nu}{I_1})(\lambda + \frac{\nu}{I_3}) - \frac{ac}{\nu^2 I_1 I_3}](\lambda + \frac{\nu}{I_2}).$$

Let $g(\lambda) = (\lambda + \frac{\nu}{I_1})(\lambda + \frac{\nu}{I_3}) - \frac{ac}{\nu^2 I_1 I_3}$, then $g(0) = \frac{1}{I_1 I_3}(\nu^2 - \frac{ac}{\nu^2}) < 0$ for $0 < \nu < \sqrt[4]{ac}$. Therefore, $g(x)$ has a positive root.

The characteristic roots of A are the two roots of $g(x)$ and $-\frac{\nu}{I_2}$, so a characteristic root of A is positive.

Let $f(t,x,y,z) = \begin{pmatrix} \frac{ayz}{I_1} \\ \frac{bxz}{I_2} \\ \frac{cxy}{I_3} \end{pmatrix}$, then $\lim\limits_{|(x,y,z)| \to 0} \frac{1}{|(x,y,z)|} f(t,x,y,z)$

$$= \lim_{(x,y,z)\to 0} \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} \frac{ayz}{I_1} \\ \frac{bxz}{I_2} \\ \frac{cxy}{I_3} \end{pmatrix} = \lim_{(x,y,z)\to 0}\begin{pmatrix} \frac{asgn(yz)}{I_1\sqrt{\frac{x^2}{y^2z^2} + \frac{1}{z^2} + \frac{1}{y^2}}} \\ \frac{bsgn(xz)}{I_2\sqrt{\frac{1}{z^2} + \frac{y^2}{x^2z^2} + \frac{1}{x^2}}} \\ \frac{csgn(xy)}{I_3\sqrt{\frac{1}{y^2} + \frac{1}{x^2} + \frac{z^2}{x^2y^2}}} \end{pmatrix} = 0.$$

Therefore,

$$f(t,x,y,z) = o(|(x,y,z)|) \qquad (|(x,y,z)| \to 0)$$

uniformly in t, $t \geq 0$.

By lemma 3.1.1, $x = 0$, $y = 0$, $z = 0$ is unstable, so $(\Omega_1^*, \Omega_2^*, \Omega_3^*) = (0, \frac{1}{\nu}, 0)$ is unstable. $\square$

I suppose that The steady states of (6), $(\Omega_1^+, \Omega_2^+, \Omega_3^+) = (\sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}, \frac{\nu}{\sqrt{ac}}, -\sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}})$ and $(\Omega_1^-, \Omega_2^-, \Omega_3^-) = (-\sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}, \frac{\nu}{\sqrt{ac}}, \sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}})$ are asymptotically stable for $0 < \nu < \sqrt[4]{ac}$. We try to prove it similarly to theorem 5.1

Let $x = \Omega_1 - \sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}$, $y = \Omega_2 - \frac{\nu}{\sqrt{ac}}$, $z = \Omega_3 + \sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}}$, then $\Omega_1 = x + \sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}$, $\Omega_2 = y + \frac{\nu}{\sqrt{ac}}$, $\Omega_3 = z - \sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}}$.

$$I_1 \frac{dx}{dt} = I_1 \frac{d\Omega_1}{dt} = a\Omega_2\Omega_3 - \nu\Omega_1 = -\nu + \sqrt{\frac{a(\sqrt{ac}-\nu^2)}{-b}}y - \sqrt{\frac{a}{c}}\nu z + ayz,$$

$$I_2 \frac{dy}{dt} = I_2 \frac{d\Omega_2}{dt} = b\Omega_1\Omega_3 - \nu\Omega_2 + 1 = -\sqrt{\frac{b(\sqrt{ac}-\nu^2)}{-a}}x - \nu y + \sqrt{\frac{b(\sqrt{ac}-\nu^2)}{-c}}z + bxz,$$

$$I_3 \frac{dz}{dt} = I_3 \frac{d\Omega_3}{dt} = c\Omega_1\Omega_2 - \nu\Omega_3 = -\sqrt{\frac{c}{a}}\nu x - \sqrt{\frac{c(\sqrt{ac}-\nu^2)}{-b}}y - \nu z + cxy.$$

Namely,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{\nu}{I_1} & \frac{1}{I_1}\sqrt{\frac{a(\sqrt{ac}-\nu^2)}{-b}} & -\frac{\nu}{I_1}\sqrt{\frac{a}{c}} \\ -\frac{1}{I_2}\sqrt{\frac{b(\sqrt{ac}-\nu^2)}{-a}} & -\frac{\nu}{I_2} & \frac{1}{I_2}\sqrt{\frac{b(\sqrt{ac}-\nu^2)}{-c}} \\ -\frac{\nu}{I_3}\sqrt{\frac{c}{a}} & -\frac{1}{I_3}\sqrt{\frac{c(\sqrt{ac}-\nu^2)}{-b}} & -\frac{\nu}{I_3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{ayz}{I_1} \\ \frac{bxz}{I_2} \\ \frac{cxy}{I_3} \end{pmatrix}.$$

Let $A = \begin{pmatrix} -\frac{\nu}{I_1} & \frac{1}{I_1}\sqrt{\frac{a(\sqrt{ac}-\nu^2)}{-b}} & -\frac{\nu}{I_1}\sqrt{\frac{a}{c}} \\ -\frac{1}{I_2}\sqrt{\frac{b(\sqrt{ac}-\nu^2)}{-a}} & -\frac{\nu}{I_2} & \frac{1}{I_2}\sqrt{\frac{b(\sqrt{ac}-\nu^2)}{-c}} \\ -\frac{\nu}{I_3}\sqrt{\frac{c}{a}} & -\frac{1}{I_3}\sqrt{\frac{c(\sqrt{ac}-\nu^2)}{-b}} & -\frac{\nu}{I_3} \end{pmatrix}$, then

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda + \frac{\nu}{I_1} & -\frac{1}{I_1}\sqrt{\frac{a(\sqrt{ac}-\nu^2)}{-b}} & \frac{\nu}{I_1}\sqrt{\frac{a}{c}} \\ \frac{1}{I_2}\sqrt{\frac{b(\sqrt{ac}-\nu^2)}{-a}} & \lambda + \frac{\nu}{I_2} & -\frac{1}{I_2}\sqrt{\frac{b(\sqrt{ac}-\nu^2)}{-c}} \\ \frac{\nu}{I_3}\sqrt{\frac{c}{a}} & \frac{1}{I_3}\sqrt{\frac{c(\sqrt{ac}-\nu^2)}{-b}} & \lambda + \frac{\nu}{I_3} \end{vmatrix} \\ &= \lambda^3 + (\frac{1}{I_1} + \frac{1}{I_2} + \frac{1}{I_3})\nu\lambda^2 + (\frac{1}{I_1 I_2} + \frac{1}{I_2 I_3})\sqrt{ac}\lambda + \frac{4\nu(\sqrt{ac}-\nu^2)}{I_1 I_2 I_3}. \end{aligned}$$

I cannot find a root of $|\lambda I - A|$, so I cannot factorize it. However, if the ODE system satisfies some symmetry, it is easy to find a root.

Theorem 6.3. *If $I_1 = I_3$, then the steady states of (6), $(\Omega_1^+, \Omega_2^+, \Omega_3^+) = (\sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}, \frac{\nu}{\sqrt{ac}}, -\sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}})$ and $(\Omega_1^-, \Omega_2^-, \Omega_3^-) = (-\sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}, \frac{\nu}{\sqrt{ac}}, \sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}})$ are asymptotically stable for $0 < \nu < \sqrt[4]{ac}$.*

Proof.

$$
\begin{aligned}
|\lambda I - A| &= \lambda^3 + (\frac{2}{I_1} + \frac{1}{I_2})\nu\lambda^2 + \frac{2\sqrt{ac}}{I_1 I_2}\lambda + \frac{4\nu(\sqrt{ac}-\nu^2)}{I_1^2 I_2} \\
&= (\lambda + \frac{2\nu}{I_1})[\lambda^2 + \frac{\nu}{I_2}\lambda + \frac{2(\sqrt{ac}-\nu^2)}{I_1 I_2}].
\end{aligned}
$$

Let $g(\lambda) = \lambda^2 + \frac{\nu}{I_2}\lambda + \frac{2(\sqrt{ac}-\nu^2)}{I_1 I_2} = (\lambda + \frac{\nu}{2I_2})^2 + \frac{2(\sqrt{ac}-\nu^2)}{I_1 I_2} - \frac{\nu^2}{4I_2^2}$, then $g(0) = \frac{2(\sqrt{ac}-\nu^2)}{I_1 I_2} > 0$.

If $\frac{2(\sqrt{ac}-\nu^2)}{I_1 I_2} - \frac{\nu^2}{4I_2^2} \leq 0$, then $g(x)$ has two negative roots;

If $\frac{2(\sqrt{ac}-\nu^2)}{I_1 I_2} - \frac{\nu^2}{4I_2^2} > 0$, then $g(x)$ has two imaginary roots with negative real part $-\frac{\nu}{2I_2}$.

The characteristic roots of A are the two roots of $g(x)$ and -2ν, so their real parts are all negative.

Let $f(t, x, y, z) = \begin{pmatrix} \frac{ayz}{I_1} \\ \frac{bxz}{I_2} \\ \frac{cxy}{I_1} \end{pmatrix}$, then similarly to theorem 6.1,

$$
f(t, x, y, z) = o(|(x, y, z)|) \qquad (|(x, y, z)| \to 0)
$$

uniformly in t, $t \geq 0$.

By lemma 3.1.1, $x = 0$, $y = 0$, $z = 0$ is asymptotically stable, so $(\Omega_1^+, \Omega_2^+, \Omega_3^+) = (\sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}, \frac{\nu}{\sqrt{ac}}, -\sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}})$ is asymptotically stable.

Similarly, $(\Omega_1^-, \Omega_2^-, \Omega_3^-) = (-\sqrt{\frac{\sqrt{ac}-\nu^2}{-bc}}, \frac{\nu}{\sqrt{ac}}, \sqrt{\frac{\sqrt{ac}-\nu^2}{-ab}})$ is also asymptotically stable. □

7 Acknowledgement

We would like to express our sincere gratitude to our advisor, Professor Theodore Drivas, for his continuous support, insightful guidance, and inspiring encouragement throughout our research and writing process.

We gratefully acknowledge Summer Math Foundation for funding this project.

We are deeply thankful to our family and friends for their unwavering support and encouragement during our studies.

Studying at Stony Brook University has been filled with pleasure and fulfilling experiences. We would like to thank the Stony Brook Department of Mathematics for providing us with high-quality education and a substantial number of opportunities, which will long benefit our future career.

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