NON-UNIQUENESS OF THREE POINT VORTEX SYSTEMS AFTER COLLAPSE

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Abstract. In the exceptional scenario of a finite-time collapse of a three point-vortex system, we demonstrate the non-uniqueness of evolution past this event. We consider a collapsing configuration with circulations given by $\Gamma_1 = 2$ $\Gamma_2 = 2$, $\Gamma_3 = -1$ and initial lengths $l_{13}(t=0) = \sqrt{6}$, $l_{12}(t=0) = 2$, $l_{23}(t=0) = \sqrt{2}$. Via the perturbation of the circulations we demonstrate that the perturbed system can evolve in one of two distinct ways: becoming either isosceles or collinear at some later time, with the outcome determined entirely by the sign of the perturbed momentum L_ϵ. For either case, a time-reversed reflexive symmetry is maintained with respect to either moment, and we distinguish the differences between both cases. We are able to demonstrate that such dichotomy directly implies the non-uniqueness of continuation past collapse.

1. Introduction

One important class of exact solutions to the Euler equations are those of point vortices, introduced by Helmholtz in 1858 [8]. A point vortex is defined by the vorticity via the Dirac delta [13],

$$\omega(R,t) = \Gamma\delta(z - z_0(t)),$$

where Γ is the circulation associated with the vortex. For a system of n-vortices, let us denote the coordinates of n-point vortices as $z_1, z_2, \dots, z_n$, where each $z_n \in \mathbb{R}^2$ is a coordinate pair $z_i = (x_i, y_i)$; each vortex has a corresponding circulation $\Gamma_i \in \mathbb{R} - \{0\}$. The vorticity is given by a superposition,

$$\omega(R,t) = \sum_i \Gamma_i \delta(z - z_i(t)).$$

A two-vortex system trivially yields a uniform rotation, however the n-vortex problem is significantly more complex. Pioneering work was done by Kirchoff [10] and Gröbli [6].

Kirchoff was the first to formulate the Hamiltonian structure of the n-vortex problem [10]. Under Kirchoff's assumptions, n-point vortices form a Hamiltonian system with three degrees of freedom; for the case of $n = 3$, the problem is completely integrable. Particularly, Synge [14] introduced trilinear coordinates for the vortex triangle and showed that the reduced problem is governed by an integrable two-dimensional system. The n-vortex problem for $n \geq 4$ is only integrable in special cases [2].

The system has three integrals of motion [13]. The Hamiltonian is given by

$$H = -\frac{1}{4\pi}\sum_{i\neq j}\Gamma_i\Gamma_j \log|z_i - z_j|. \tag{1}$$

It isn't hard to notice that H only depends on circulations and relative distances, i.e., it is invariant under rotations and translations. That, of course, implies conservation of linear and angular momenta:

$$P \equiv \sum_i \Gamma_i z_i(t) = \text{const.},$$
$$L \equiv \sum_i \Gamma_i z_i^2(t) = \text{const.}$$

The equations of motion describing the evolution of the system are

$$\begin{aligned}\Gamma_i \frac{d}{dt}x_i &= \frac{\partial}{\partial y_i}H,\\ \Gamma_i \frac{d}{dt}y_i &= -\frac{\partial}{\partial x_i}H,\end{aligned} \tag{2}$$

Remarkably, this integrable three-vortex system can exhibit finite-time singularities. Gröbli provided a detailed analysis of three-point vortex systems in his dissertation [6]. Perhaps the most important result was the finite-time collapse of such systems. In a collapsing solution the vortex triangle is self-similar, so that the vortices coalesce together at a finite collapse time. Finite time collapse has been studied in relative detail, including requirements for collapse [1, 9].

Proposition 1. *A three point-vortex configuration collapses if and only if* [13]

$$\begin{cases} L = 0 \\ \sum_{ij}\Gamma_i\Gamma_j = 0. \end{cases}$$

Proof. Clearly, L must vanish for collapse to occur. Recall that if collapse occurs it must be self similar[13], and the Hamiltonian given by Eq. (1) is a constant of motion. Then, denoting the length between vortices z_i and z_j as l_{ij},

$$l_{ij}(t) = \lambda(t) l_{ij}(0).$$

Then,

$$H(t=0) = \frac{-1}{4\pi}\sum_{ij}\Gamma_i\Gamma_j \log l_{ij}(0) = \frac{-1}{4\pi}\sum_{ij}\Gamma_i\Gamma_j \log(\lambda(t) l_{ij}(0)) = H(t=0) + \sum_{ij}\Gamma_i\Gamma_j \log(\lambda(t)).$$

H being constant implies

$$\sum_{ij}\Gamma_i\Gamma_j = 0,$$

as desired. □

Collapse is exceptionally non-generic — that is, it is a measure-zero event [5]. Synge [14] found that a harmonic mean-like quantity,

$$\Gamma = \Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3,$$

is a primary parameter to classify configurations, where

$$\Gamma > 0: \text{ elliptic},\ \Gamma = 0: \text{parabolic},\ \Gamma < 0: \text{hyperbolic}.$$

Because collapse is such a rare event, as well as due to the singularity present at collapse, several works have examined the dynamics of point vortex systems near, but not exactly at, collapse. Of particular interest to us is the study of near-collapse behavior by Leoncini et al. [12], who analyzed the evolution of three-point vortex systems initialized close to collapsing configurations. By reformulating the problem in terms of an effective potential, they showed that when collapse conditions are slightly violated, the system does not collapse in finite time but instead exhibits extremely slow, tightly bound motion. Depending on the nature of the perturbation, the characteristic time scale for the vortex triangle to escape the near-collapsing region can diverge in various ways with respect to the perturbation amplitude, such as via power-law or logarithmic divergence.

Grotto and Pappalettera [7] extended this line of inquiry by constructing weak solutions to the two-dimensional Euler equations that exhibit “bursts”, a reverse collapse phenomenon in which three vortices emerge from a singular point. Their construction shows that, in the weak sense of vorticity measures, it is possible to have non-unique forward evolutions after such a collapse-like event.

In this paper, we aim to build on the ideas of continuation past collapse, particularly of non-uniqueness. Starting from an exactly collapsing configuration of three vortices, we introduce small perturbations of the circulations resulting in a nonzero momentum L_ϵ. This leads to two distinct families of solutions near the time of collapse; in particular, we show that the sign of L_ϵ determines whether the perturbed system becomes collinear or isosceles at some later time, with each scenario exhibiting time-reversal and reflexive symmetry. Thus, while the unperturbed collapsing system formally leads to a singularity at finite time, its perturbed variants suggest two qualitatively different continuations. Our work extends the analysis of collapse by identifying and characterizing the branching structure of its near-singular continuations.

2. Results

We again consider the collapsing configuration with circulations given by

$$\Gamma_1^c = 2,\ \Gamma_2^c = 2,\ \Gamma_3^c = -1,$$

and initial lengths

$$l_{12}(t=0) = 2,\ l_{13}(t=0) = \sqrt{6},\ l_{23}(t=0) = \sqrt{2},$$

where l_{ij} denotes the length between the i^{th} and j^{th} vortex positions, z_i and z_j. We perturb the circulations of this configuration by some $(\epsilon_1, \epsilon_2, \epsilon_3)$, such that a new pair triple of circulations given by

$$\Gamma_i = \Gamma_i^c + \epsilon_i,$$

$i \in \{1, 2, 3\}$ and the same initial lengths. Clearly any such configuration no longer collapses. For small choices of ϵ, we call such non-collapsing configurations $\epsilon-$perturbed configurations, and we let L_ϵ and H_ϵ be the consequent constants of motion.

We claim, that for any such $\epsilon-$perturbed system there are only two possibilities: the vortices either become collinear, or form an isosceles triangle at some moment in time. However for times before either of those moments, the motion of $\epsilon-$perturbed configurations aligns with the motion of the collapsing configuration for $t < t^c$ [3], where t^c denotes the time of collapse of the original system. Due to this, it becomes apparent that the area of the $\epsilon-$perturbed configuration must reach a minima near $t = t^c$. We can study what happens to the motion of the vortices in this scenario. Indeed, if the vortices are ever aligned, the area is zero. In fact, the motion of the point-vortex system is symmetric about the line of collinearity.

Lemma 1. *If a $3-P.V$ configuration becomes collinear at $t = t^*$, then the trajectories traced out by the point vortices are symmetric about the collinearity line formed by the vortex motion.*

Proof. Consider the vector $(z_1, z_2, z_3) \in \mathbb{R}^6$, which travels via the action of the autonomous vector field

$$\left(\frac{\partial H}{\partial y_1}, -\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial y_2}, -\frac{\partial H}{\partial x_2}, \frac{\partial H}{\partial y_2}, -\frac{\partial H}{\partial x_2}\right).$$

Assume, without loss of generality, that the line of collinearity is $x = 0$. The transformation of reflection across this line:

$$R : (x_1, y_1, \dots, x_3, y_3) \mapsto (-x_1, y_1, \dots, -x_3, y_3). \tag{3}$$

Notice, that the Hamiltonian in Eq. (1) only depends on relative distances $|z_i - z_j|$, so it is invariant under the transformation.

Consider now the time reversal operator

$$\Theta : t \mapsto -t.$$

Let us apply the transformation $\tilde{z}(t) \coloneqq [R\Theta]z(t)$, i.e., we reflect and time reverse the trajectories (specifically, we're interested for $t < t^*$). Then,

$$\tilde{x}(t) = -x(-t),\ \tilde{y}(t) = y(-t).$$

Via the chain rule,

$$\begin{aligned}\frac{\partial H}{\partial \tilde{x}_i}(z(t)) &= -\frac{\partial H}{\partial x_i}(z(-t)),\\ \frac{\partial H}{\partial \tilde{y}_i}(z(t)) &= \frac{\partial H}{\partial y_i}(z(-t)).\end{aligned}$$

The time derivatives become

$$\dot{\tilde{x}}_i(t) = \dot{x}_i(-t),\ \dot{\tilde{y}}_i(t) = -\dot{y}_i(-t).$$

From the original equations of motion 2,

$$\begin{aligned}\Gamma_i \frac{d}{dt} x_i(-t) &= \frac{\partial}{\partial y_i} H(z(-t)),\\ \Gamma_i \frac{d}{dt} y_i(-t) &= -\frac{\partial}{\partial x_i} H(z(-t)).\end{aligned}$$

It is therefore clear that the trajectories $\tilde{z}(t)$ satisfy the same equations of motion, i.e.,

$$\Gamma_i \dot{\tilde{x}}_i = \frac{\partial H}{\partial \tilde{y}_i},\ \Gamma_i \dot{\tilde{y}}_i = -\frac{\partial H}{\partial \tilde{x}_i}.$$

By original hypothesis, the original trajectory becomes symmetric at $t = t^*$. That is, we have

$$\tilde{z}(t) = R(z(2t^* - t)),$$

so that $\tilde{z}(t^*) = z(t^*)$. Both $t \mapsto z(t)$ and $t \mapsto \tilde{z}(t)$ are solutions to the same autonomous differential equation, and they coincide at the time $t = t^*$. By the Picard–Lindelöf theorem [4], they must coincide for all t. Hence, the forward trajectory for $t > t^*$ is equivalent to the reflected backward trajectory for $t < t^*$. □

Corollary 1. *In the special case of $\Gamma_1 = \Gamma_2$, if the triangle does not cross itself, the system retains a reflexive symmetry in terms of the triangle itself, without the relabeling of vortex positions.*

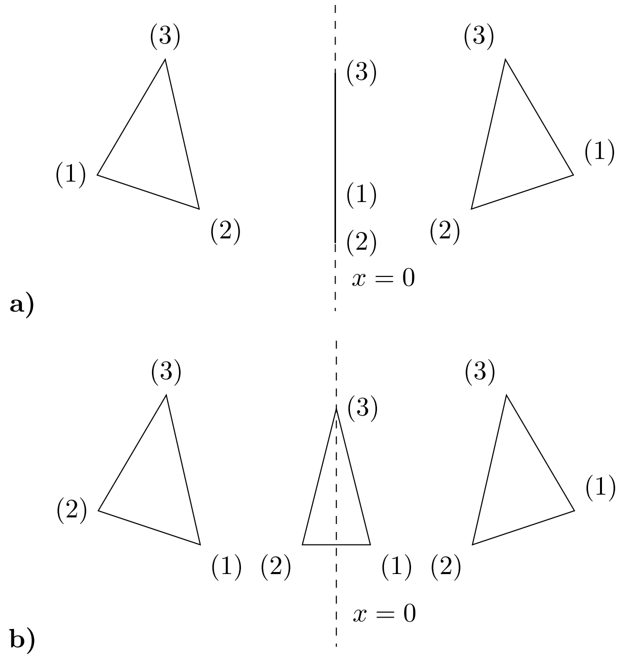

FIGURE 1. (a) True reflection. For such a reflection, the triangle must cross itself, i.e., become collinear. (b) No reflcetion, and vortices do not permute.

Proof. Suppose we have a situation in which a reflection does not occur (see, e.g., Fig. 1b). Let us write a transformation

$$P : (x_1, y_1, x_2, y_2, x_3, y_3) \mapsto (-x_2, y_1, -x_1, y_2, -x_3, y_3).$$

This represents a symmetry where two of the vortices swap.

Note that the equations of motion in Eq. (2) depend only on circulations and relative distances. That is, if $\Gamma_i = \Gamma_j$, a permutation $i \leftrightarrow j$ does not affect the dynamics of the system.[1] Thus, if $\Gamma_1 = \Gamma_2$, the transformation above can be written as

$$P_{\text{eff}} : (x_1, y_1, \ldots, x_3, y_3) \mapsto (-x_1, y_1, \ldots, -x_3, y_3),$$

which we notice is the exact same transformation as the reflection in Fig. 3. Hence, if $\Gamma_1 = \Gamma_2$, there is an effective reflection $P_{\text{eff}} = R$ even if the triangle does not cross itself yet still contains such a symmetry. □

Even if the configuration does not reflect in the usual way via a bijection between the vortices, it still essentially reflects the actual motion of the triangle, leaving the actual roles of the vortices untouched. Hence, we have the following corollaries, in terms of a $\epsilon-$perturbed collapsing configuration:

Corollary 2. *Suppose for $t = t^*$, the vortices of the $\epsilon-$configuration are collinear. Then, after t^*, the vortices permute their roles in the triangle, and the orientation of the triangle formed by the $\epsilon-$perturbed system is reversed.*

[1]The only difference is the sign of the area.

Corollary 3. *Suppose for $t = t^*$, the vortices form an isosceles triangle up to an $\epsilon-$approximation.*[2] *Then, after t^*, the roles of the vortices in the triangle remains unchanged.*

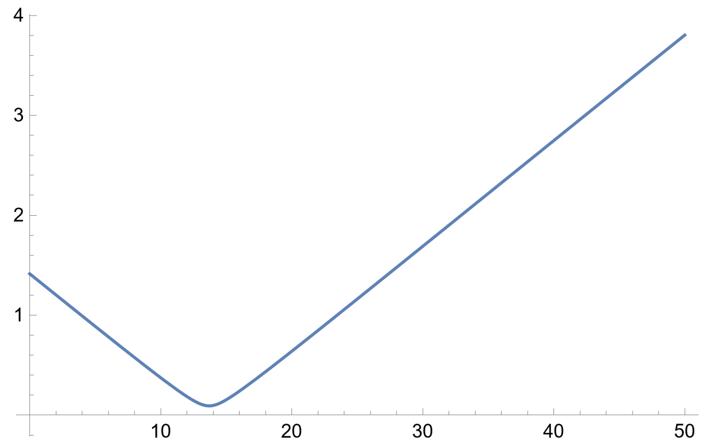

FIGURE 2. Signed area, $A(t)$, of the $\epsilon-$perturbed configuration with $\epsilon-$perturbation $(\epsilon_1, \epsilon_2, \epsilon_3)$ such that $L_\epsilon > 0$.

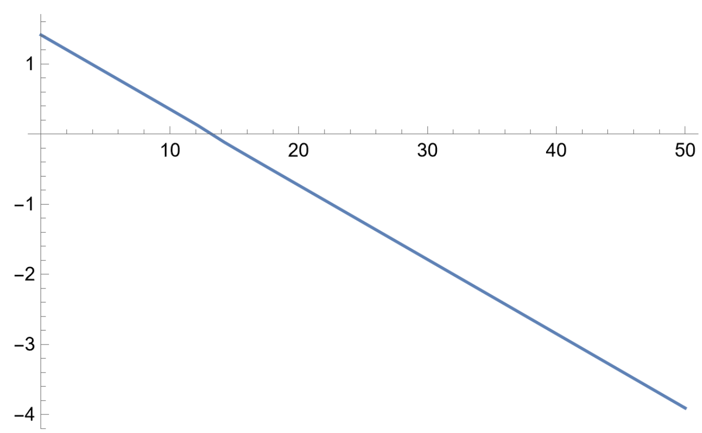

FIGURE 3. Signed area, $A(t)$, of the $\epsilon-$perturbed configuration with $\epsilon-$perturbation $(\epsilon_1, \epsilon_2, \epsilon_3)$ such that $L_\epsilon < 0$.

If the collapsing configuration with circulations $\Gamma_1^c = 2$, $\Gamma_2^c = 2$, $\Gamma_3^c = -1$ is perturbed, we demonstrate that the mirror symmetry in the system emerges from the moment of minimal area of the three vortex system. Thus, we consider the following proposition:

[2]We say ϵ-approximation because in this case, the circulations Γ_1 and Γ_2 are not exactly identical but rather are within some small ϵ of each other.

Proposition 2. *If $(\epsilon_1, \epsilon_2, \epsilon_3)$ perturbs the collapsing configuration with circulations $(\Gamma_1^c, \Gamma_2^c, \Gamma_3^c) = (2, 2, -1)$, then the following cases are possible.*
1. If $(\epsilon_1, \epsilon_2, \epsilon_3)$ is such that $L_\epsilon > 0$, then there exists $t = t^*$ such that the $\epsilon-$perturbed system forms an isosceles triangle, up to an $\epsilon-$approximation.
2. If $(\epsilon_1, \epsilon_2, \epsilon_3)$ is such that $L_\epsilon < 0$, then $\frac{dA}{dt} < 0$ for all t, and $A(t) \to -\infty$ as $t \to \infty$.

Consider the case such that $(\epsilon_1, \epsilon_2, \epsilon_3)$ is such that $L_\epsilon > 0$. First, we rule out that collinearity cannot occur for any t for such choices of ϵ_i, which can also be seen in Figure **??**.

Lemma 2. *If $L_\epsilon > 0$, then the $\epsilon-$perturbed configuration does not become collinear for any time t.*

Proof. Suppose $(\epsilon, 0, 0)$ is the change in the collapsing circulations $(2, 2, -1)$ such that $L_\epsilon > 0$. Assume towards contradiction that the $\epsilon-$perturbed configuration becomes collinear at some time $t = t^*$, for which

$$\begin{cases} l_{13} = \pm l_{12} \pm l_{23} \\ l_{12} = \lambda l_{23} \end{cases}, \tag{4}$$

$\lambda \in \mathbb{R}$. Then we can rewrite

$$L_\epsilon = \Gamma_1\Gamma_2 l_{23}^2 + \Gamma_1\Gamma_3 l_{13}^2 + \Gamma_2\Gamma_3 l_{23}^2 = l_{23}^2(\Gamma_1\Gamma_2\lambda^2 + \Gamma_1\Gamma_3(\lambda \pm 1)^2 + \Gamma_2\Gamma_3) =: l_{23}^2 P_\pm^\epsilon(\lambda),$$

which is true always as L is a constant of motion. $P_\pm^\epsilon(\lambda)$ is the quadratic polynomial with coefficients $\epsilon-$close to $P(\lambda)_\pm = 2\lambda^2 \mp 4\lambda - 4$. The roots of $P_+(\lambda)$ are $1 \pm \sqrt{3}$, and the roots of $P_-(\lambda)$ are $-1 \pm \sqrt{3}$. First we suppose

$$\begin{cases} l_{13} = l_{12} + l_{23} \\ l_{12} = \lambda l_{23} \end{cases},$$

in which case $L_\epsilon > 0$ whenever $\lambda \in (-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$. Then, H at $t = t^*$ can be rewritten as

$$H_\lambda = \frac{-1}{4\pi}[4\log|\lambda| - 2\log|\lambda+1| - \epsilon(2\log|\lambda l_{23}| + \log|l_{23}(\lambda+1)|].$$

However, we have that

$$H_c = -0.022893$$

at the collapsing configuration. So, for small enough perturbations, H_ϵ will still be very close to the value of -0.022893. However, if we consider H_λ on $(-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$ we see that the maximum value is at $C = -0.10425$, but

$$C < -0.022893,$$

which is a contradiction since H_ϵ is constant for all t.

Now, we consider the case

$$\begin{cases} l_{13} = l_{12} - l_{23} \\ l_{12} = \lambda l_{23} \end{cases}$$

for $t = t^*$, and we consider $\lambda \in (-\infty, -1-\sqrt{3}) \cup (-1+\sqrt{3}, \infty)$.
First let us suppose $\lambda \in (-1+\sqrt{3}, \infty)$. Then, $\lambda - 1 > 1$ and hence, we can rewrite

$$l_{12} = \frac{\lambda}{\lambda - 1} l_{13}$$

in which case, the $\epsilon-$configuration must have already passed an isosceles configuration for $t < t^*$, by the original ordering of the lengths being

$$l_{13}(0) > l_{12}(0) > l_{23}(0).$$

Now suppose $\lambda \in (-\infty, -1-\sqrt{3})$. Then,

$$l_{13} = l_{23}(\lambda - 1) < 0,$$

which has no physical meaning in regards to length between two vortices. Now, suppose that

$$\begin{cases} l_{13} = -l_{12} + l_{23} \\ l_{12} = \lambda l_{23} \end{cases},$$

for $\lambda \in (-\infty, -1-\sqrt{3}) \cup (-1+\sqrt{3}, \infty)$.

We rewrite H_λ at $t = t^*$ as

$$H_\lambda = \frac{-1}{4\pi}[4\log|\lambda| - 2\log|-\lambda+1| - \epsilon(2\log|\lambda l_{23}| + \log|l_{23}(-\lambda+1)|)].$$

However $H|_{(-\infty,-1-\sqrt{3})\cup(-1+\sqrt{3},\infty)}$ attains a maximum value of -0.11, which cannot happen to similar reasoning as in case I. Lastly, suppose

$$\begin{cases} l_{13} = -l_{12} - l_{23} \\ l_{12} = \lambda l_{23} \end{cases},$$

for $\lambda \in (-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$. If $\lambda \in (1+\sqrt{3}, \infty)$, similar to case II, l_{13} would not be physically possible. If $\lambda \in (-\infty, 1-\sqrt{3})$, the $\epsilon-$configuration would have to pass through an isosceles configuration previously, as shown in case II. ☐

Since for $L_\epsilon > 0$ the $\epsilon-$perturbed system cannot become collinear, we study other possibilities for when the configuration attains a minimum value.

Lemma 3. *Assume $(\epsilon_1, \epsilon_2, \epsilon_3)$ are such that $L_\epsilon > 0$. Then,*

$$\frac{dA}{dt} = 0 \tag{5}$$

at some $t = t^*$ if and only if the $\epsilon-$perturbed configuration is isosceles up to an $\epsilon-$approximation at t^*.

Proof. Recall that $dA(t)/dt$ is given by [11]

$$\frac{dA}{dt} = \frac{1}{8\pi}\left(4\frac{l_{23}^2 - l_{13}^2}{l_{12}^2} + \frac{l_{13}^2 - l_{12}^2}{l_{23}^2} + \frac{l_{12}^2 - l_{23}^2}{l_{13}^2}\right), \tag{6}$$

in which case the leftward direction is automatic. Suppose we consider a $\epsilon-$perturbed configuration with circulations given by $(2+\epsilon, 2, -1)$. Then,

$$H_\epsilon = \frac{-1}{4\pi}[(2+\epsilon)2\log(l_{12}(t)) - (2+\epsilon)\log(l_{13}(t)) - 2\log(l_{23}(t))] = -0.022893 + \epsilon\left(2\log\sqrt{2} - \log\sqrt{6}\right),$$

which equivalently, gives the following relationship between the side lengths:

$$\frac{l_{12}^{2(2+\epsilon)}}{l_{13}^{2+\epsilon} l_{23}^2} = e^{4\pi(0.022893+\epsilon(0.2027))},$$

which we rewrite as

$$l_{12}^4 = l_{13}^2 l_{23}^2 e^{4\pi 0.022893}\left(\frac{l_{13}^{\epsilon}}{l_{12}^{2\epsilon}} e^{4*0.02027\epsilon}\right).$$

Let δ_ϵ be such that

$$\frac{l_{13}^{\epsilon}}{l_{12}^{2\epsilon}} e^{4*0.02027\epsilon} = \delta_\epsilon(t).$$

and we let $l_{12}(t) = l_{13}(t) - C(t)$, where $C(t) \in \mathbb{R}$. Suppose $t = t'$ is such that $\frac{l_{13}(t)}{l_{13}(t)-C(t)}$ is at a maximum. Then we can bound δ_ϵ,

$$e^{4*0.02027\epsilon} \leq \delta_\epsilon(t) \leq \left(\frac{l_{13}(t')}{(l_{13}(t') - C(t'))^2} e^{4*0.02027}\right)^{\epsilon}.$$

As collapse no longer occurs in the perturbed system, the upper bound on δ_ϵ is continuous everywhere. So if we only consider δ_ϵ as a function of ϵ, then for small choices of ϵ,

$$\frac{dA_\epsilon}{dt} = \frac{1}{8\pi}\left[(4+\epsilon)\frac{l_{23}^2 - l_{13}^2}{e^{2\pi(0.022893)} l_{13} l_{23}\delta_\epsilon} + \frac{l_{13}^2 - e^{2\pi(0.022893)} l_{13} l_{23}\delta_\epsilon}{l_{23}^2} + (1+\epsilon)\frac{l_{13} l_{23} e^{2\pi(0.022893)}\delta_\epsilon - l_{23}^2}{l_{13}^2}\right] = 0$$

gives us solutions $l_{13} = \pm l_{23}$, $l_{13} = \frac{2.05}{9} l_{23}$, $l_{13} = \frac{9}{2.05} l_{23}$, up to an $\epsilon-$approximation. $l_{13} = -l_{23}$ denotes an isosceles solution, however has no physical meaning, so we omit it. If for some $t = t^*$

$$l_{13} = \frac{2.05}{9} l_{23},$$

we can see that the perturbed configuration must have passed through an isosceles configuration for some $t < t^*$, since $l_{13}(0) > l_{23}(0)$, so this case can be ruled out. If for some $t = t^*$

$$l_{13} = \frac{9}{2.05} l_{23},$$

we compute

$$L_\epsilon = l_{23}^2\left(\frac{(4)(9)e^{2\pi 0.022893}}{2.05} - \frac{(2)(9^2)}{2.05^2} - 2\right) + \epsilon l_{23}^2\left(\frac{(2)(9)}{2.05} e^{2\pi 0.022893} - \frac{9^2}{2.05^2}\right) < 0$$

for any choice of ϵ, which contradicts the assumption. Now, suppose

$$l_{13} = l_{23}$$

for $t = t^*$. Then,

$$L_\epsilon = l_{23}^2(4e^{2\pi 0.022893} - 4) + \epsilon l_{23}^2(2e^{2\pi 0.022893} - 1),$$

which is always positive for $|\epsilon|$ being sufficiently small. So, we were able to omit any of the other possible solutions, and be left with the isosceles solutions. Since the motion of the is close to the motion of the $\epsilon-$perturbed configuration for $t < t^c$, but the $\epsilon-$perturbed configuration but avoids collapse, it must reach a minimum value at some point, and we showed the only possibility is the isosceles case. □

Due to the mirror symmetry that follows a isosceles by Corollary 1, we are able to rule out the case of dA/dt not changing sign.

Now let us suppose we choose $(\epsilon_1, \epsilon_2, \epsilon_3)$ such that $L_\epsilon < 0$. Then, we can show the following.

Lemma 4. *Consider the collapsing configuration with circulations (2,2-1), and suppose we choose an $\epsilon-$perturbation such that $L_\epsilon < 0$. Then,*

$$\frac{dA(t)}{dt} < 0$$

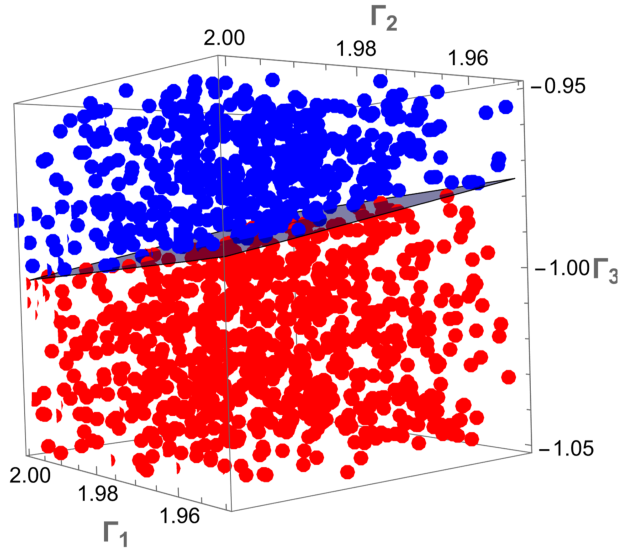

FIGURE 4. Distribution of $\epsilon-$perturbed configurations in a $3-$space of circulations. Red points represent perturbed configurations that become collinear; blue points represent configurations that do not. We see a clear separation between these points, defined by the surface $L = 0$.

for all t.

Proof. Suppose that $L_\epsilon < 0$. We can show that for $L_\epsilon < 0$, the area in time does not change sign. In particular, we can show

$$\frac{dA}{dt} = 0$$

does not occur for any time t^*. Assume that for the $\epsilon-$perturbation such that $L_\epsilon < 0$, the three point vortices form a triangle for any time t. Then there must exist a time for which area reaches a minima, either absolutely or asymptotically. In particular, let us consider all the possible solutions when $dA/dt = 0$. Immediately we rule out the case of $l_{13} = l_{23}$, because this solution contradicts the sign of L_ϵ, as per Lemma 3. Similarly, we rule out the $l_{13} = \frac{2.05}{9} l_{23}$ as an isosceles configuration is attained previously by the system in this scenario. Now suppose for some t^*,

$$l_{13} = \frac{9}{2.05} l_{23}.$$

However,

$$l_{23} + l_{12} = l_{23}(1 + \frac{9^{1/2}}{2.05^{1/2}} e^{\pi 0.022893}) < \frac{9}{2.05} l_{23} = l_{13},$$

contradicting the triangle inequality. □

Indeed, we now showed that for $L_\epsilon < 0$, dA/dt does not change sign. In fact,

$$A(t) \to -\infty$$

as $t \to \infty$. Otherwise,

$$\frac{dA(t)}{dt} \to 0$$

for t large enough, but that means for such values of t, $L_\epsilon > 0$. But, since L_ϵ is a constant of motion, $L_\epsilon > 0$ for all t. So, if $L_\epsilon < 0$,

$$\frac{dA(t)}{dt} < 0$$

and,

$$A(t) \to -\infty.$$

The case of $L_\epsilon < 0$ can be seen in Figure 3.

Therefore we showed that Proposition 2 holds. In particular, we showed that based on the sign of the $\epsilon-$perturbation, we get two possibilities from the perturbed configuration. And hence, a separation arises between the set of $\epsilon-$configurations that become isosceles at some time, and the set of those configurations that become collinear; they are separated by the surface $L = 0$, as seen in Figure 4. Such property can directly provide insight to the possibility of non-uniqueness after collapse of our original system. Let us consider a sequence of $\epsilon-$perturbations, $\{\epsilon_i\}_i$ converging to 0 such that

$$L_{\epsilon_i} < 0$$

for each i, and let $\{\epsilon'_j\}_j$ be another sequence of $\epsilon-$perturbations that converges to zero, and

$$L_{\epsilon'_j} > 0$$

for each j. Then, by Proposition 2, for $i, j > N_{i,j}$, two possibilities can occur: that is, either the roles of the vortices in the triangle are permuted and the orientation of the triangle changes; or the roles of the vortices in the triangles remain the same, hence implying non-uniqueness after the event of finite-time collapse. This suggests the non-uniqueness of Euler equations near finite-time singularities.

Acknowledgments

We would like to thank our advisor Professor Theodore Drivas, as well as Daniil Glukhovskiy for their continuous insight, encouragement, and support. We also thank our friend Boris Khanikati for thoughtful discussions, and moral support. We are grateful for the Summer Math Foundation for funding this project.

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