Utilization of Bifurcation Diagrams and Lyapunov Exponent Graphs to Mathematically Model Noise-Induced Order

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Abstract

Noise-induced order is a phenomenon in which noise is added to a chaotic system, and order results. Noise induced order was originally identified in the Belousov-Zhabotinsky oscillating chemical reaction, specifically as characterized by the Matsumoto-Tsuda model. A major question in the field is the search for models that display noise induced order, in addition to the Matsumoto-Tsuda model. In this study, noise perturbations added to the distinct bifurcation diagrams were gradually increased in magnitude (numbers of intervals) until either the Lyapunov exponent became negative throughout the map, or it became clear that this would not be the case. The Lyapunov exponent indicates chaos when it is positive, and order when it is negative. The results of this study confirm that the Matsumoto-Tsuda model does indeed display noise-induced order, however, the results also show some noise induced order in models that had larger regions of flat curves or curves with more regions of derivatives of smaller magnitude. Induced order has been seen to have major implications in health, and so this data could be valuable for future research.

Introduction

Chaos Theory:

Oftentimes in scientific and mathematical equations and formulas, an idealized situation is needed in order for the calculations to hold true. Rarely do these equations accurately represent real life situations. Reality has many factors that can alter or change a situation and its outcome. This is why chaos theory has so much potential - it is a way to represent real life situations. Chaos theory is a branch in mathematics focusing on the study of dynamical systems that seem to display disorder, or randomness. However, contrary to its appearance, chaos is actually nonrandom, and has complete determinism. It is extremely sensitive and dependent on initial conditions. Known as the Butterfly effect, extremely small changes in the initial condition can significantly alter the results. When a system moves into chaos, it is actually only an extremely rapid increase in errors that, though governed by certain conditions, are extremely difficult, if not impossible, to predict long term.

The graph depicted below in figure 1a is a bifurcation diagram of the logistic model ($f(x) = rx(1-x)$), the standard and most commonly known bifurcation diagram in chaos theory. It is an extremely clear visual representation of a system moving into chaos. This graph plots the values that the system oscillates between over time. Initially, there is period doubling, which is predictable, however as time goes on, the behavior becomes chaotic. In figure 1b, the wide spread of black points on the right side is indicative of chaotic behavior, as the system is erratically jumping between many, unpredictable values. The sections with white space throughout the chaos represent short periods of order, which is characteristic of most bifurcation diagrams.

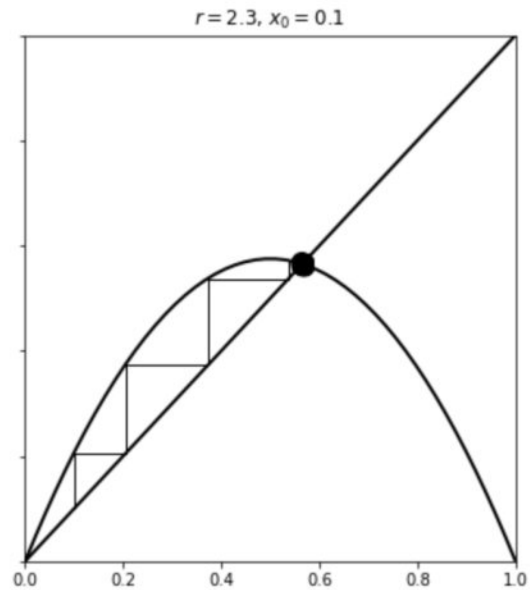

Figure 1a

Logistic Map (r = 2.3)

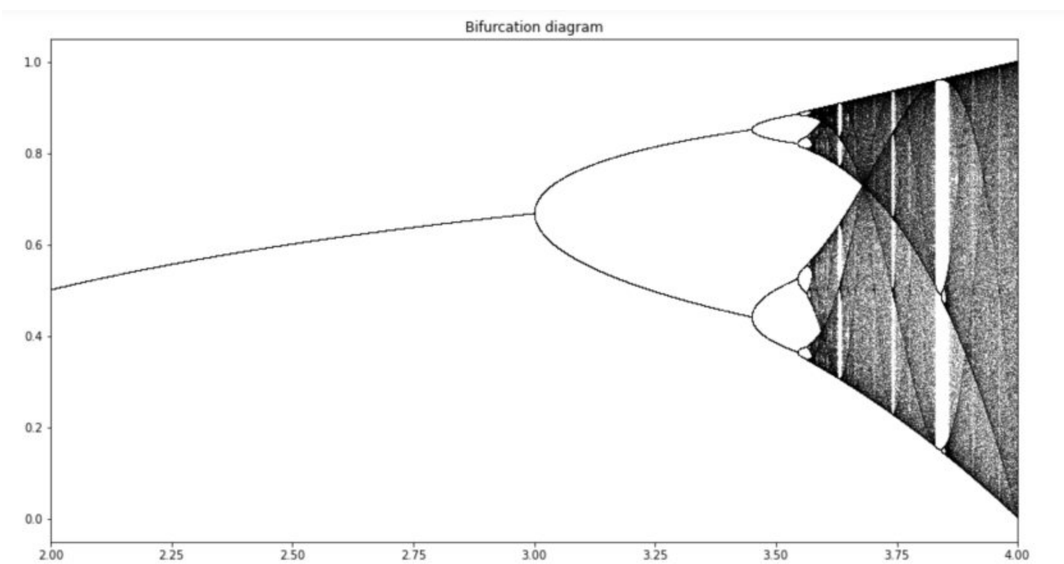

Figure 1b

Logistic Map Bifurcation Diagram

Lyapunov Exponent:

The Lyapunov Exponent is a tool used to measure the rate of divergence of nearby trajectories, or in this case, measure the degree of chaotic behavior. When the Lyapunov Exponent is positive, that means chaotic behavior is present. In fact, a positive Lyapunov Exponent essentially defines chaos. The more positive the value is, the more chaotic the system. However, when the Lyapunov exponent is negative, that means that there is order at that value, and chaos is no longer present.

In a previous study done on Belousov Zhabotinsky reactions (BZ reactions for short), shown in figure 3, by S. Galatolo, M. Monge, and I. Nisoli, noise was used to stabilize the chaos in its behavior, or induce order. By adding noise perturbations to each iteration of the return map in figure 2 of a BZ function (given below), the chaotic behavior in the resulting bifurcation diagram gradually disappeared as noise levels were increased, and the Lyapunov exponent became negative throughout.

$$T_{a,b,c}(x) = \begin{cases} (a + (x - \frac{1}{8})^{\frac{1}{3}})e^{-x} + b & 0 \leq x \leq .3 \\ c(10xe^{\frac{-10x}{3}})^{19} + b & .3 < x \leq 1 \end{cases}$$

This function, known as the Matsumoto-Tsuda model of BZ reactions, maps the interval [0,1] to the interval [0,1]. The values of a, b, and c fall on extremely, extremely small intervals such that they are essentially constants. A figure of the model can be found below. There is a region containing a hump as well as essentially flat regions.

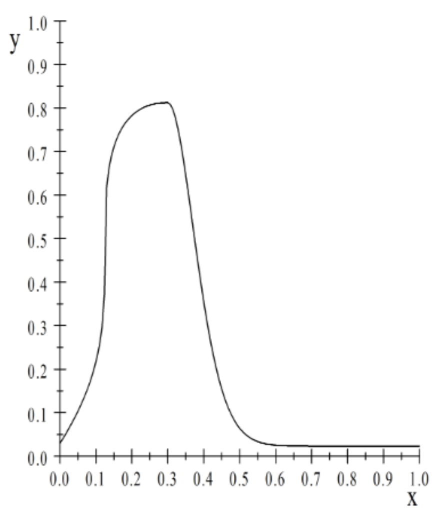

Figure 2

Matsumoto-Tsuda Model (Galatolo et al.)

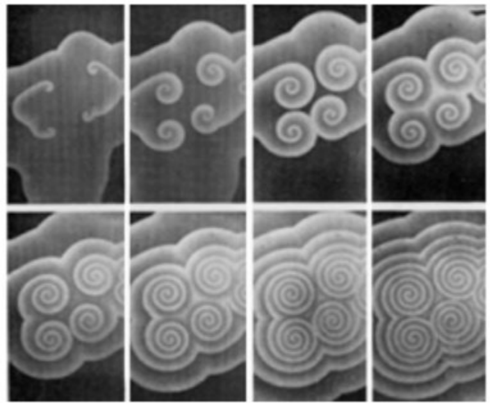

Figure 3

Belousov-Zhabotinsky Reaction (Galatolo et al.)

By taking the bifurcation diagram of the Matsumoto-Tsuda model, other BZ reaction maps as well as the logistic map, corresponding Lyapunov Exponent graphs can be made. In this fashion, chaotic behavior can be observed as noise is increased.

Methods:

In order to confirm and graph the noise induced order within the Matsumoto-Tsuda model, Python was used to map out the iterations, create the bifurcation diagram, as well as plot the Lyapunov exponent. Usually, the Lyapunov exponent is calculated using the given function:

$$\lambda_\xi := \int_0^1 \log |T'(x)| \, d\mu_\xi$$

where ξ is the noise size that is kept within the bounds $\left[\frac{8.73}{10^5}, 1/2\right]$ and μ_ξ is the unique ergodic absolutely continuous stationary measure. However, the equation was slightly altered to:

$$\tilde{\lambda}(\varepsilon) = \frac{1}{N} \sum_{i=1}^{N} ln(T_\varepsilon{}'(x_i))$$

which is equivalent by ergodicity to the previous equation in terms of the invariant measure.

Then, the same process was used to create bifurcation diagrams and graphs for the logistic model as well as a different Belousov Zhabotinsky Reaction model representing the high-flow-rate-chaotic state. The graph for the high-flow-rate-chaotic state was first traced and then the bifurcation diagram and the Lyapunov exponent diagram was made. Noise was gradually added and increased in these diagrams in order to see whether or not an ordered state could be reached.

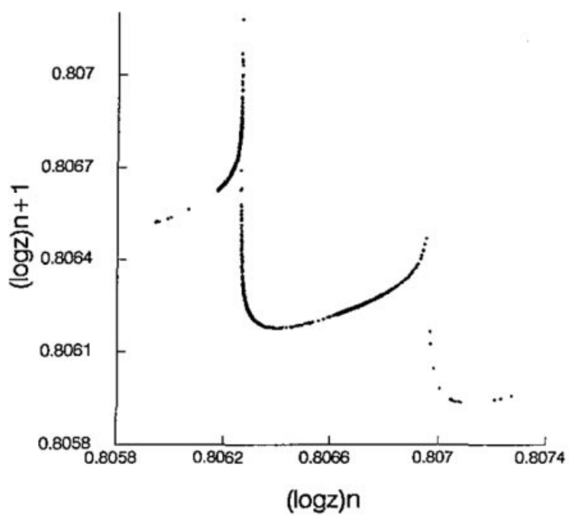

Figure 4

Belousov Zhabotinsky Reaction High-Flow-Rate-Chaotic State Data (Zhang et al.)

Noise perturbations were added to these systems by assigning noise a value that would then be multiplied to generated values within a certain range and then this noise would be added to the iterations in the bifurcations diagram. As the noise value increased, the numbers calculated when multiplying would be larger, resulting in greater perturbations and changes on the original values. In this fashion, the noise value was increased until the lyapunov exponent graph became consistently negative throughout, meaning that all chaos had been removed from the system, or until it became clear that the lyapunov exponent would never settle at complete order.

Results

Belousov Zhabotinsky Reaction Matsumoto-Tsuda model:

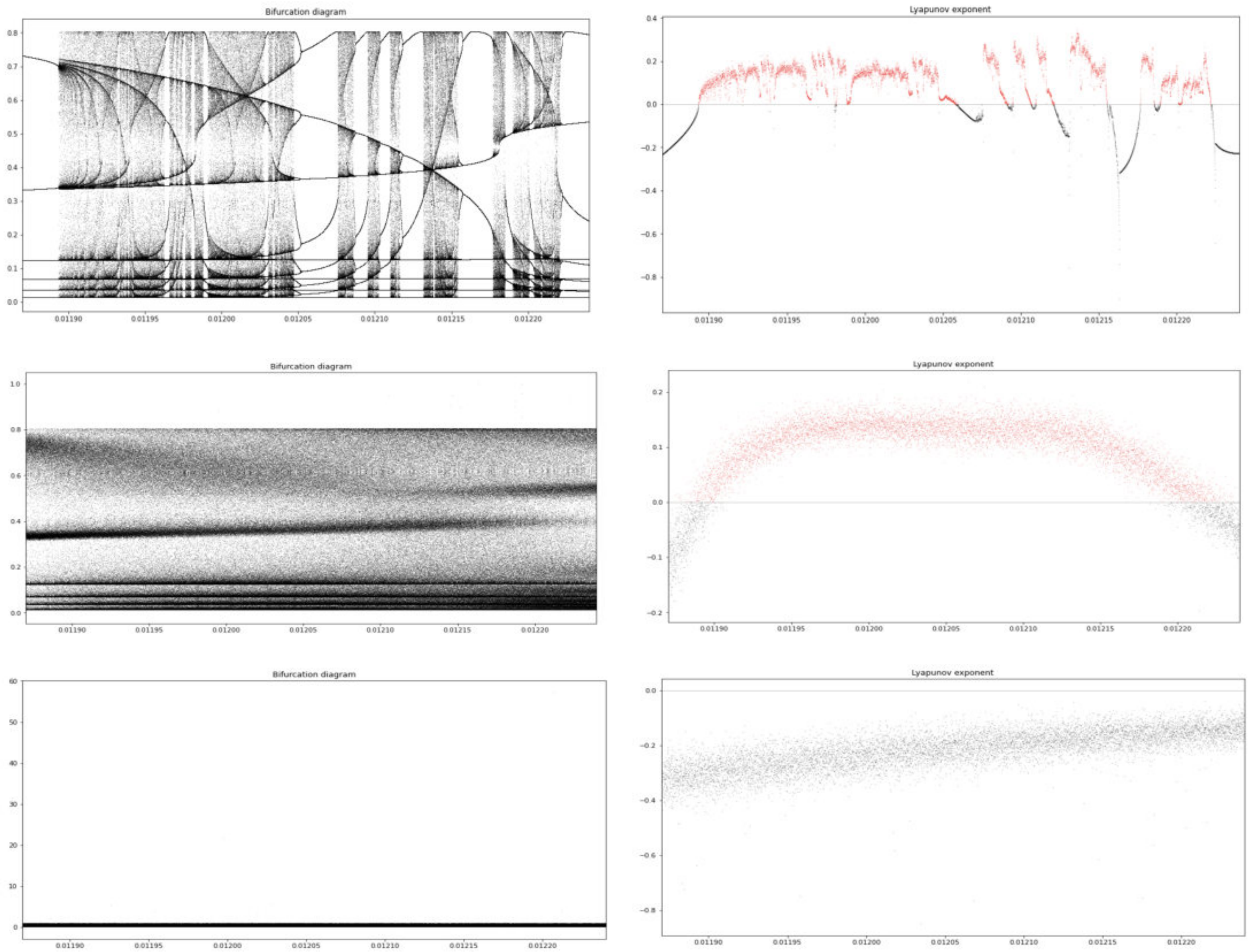

Figure 5

Matsumoto-Tsuda model: bifurcation diagram and corresponding lyapunov exponents graph

Logistic model:

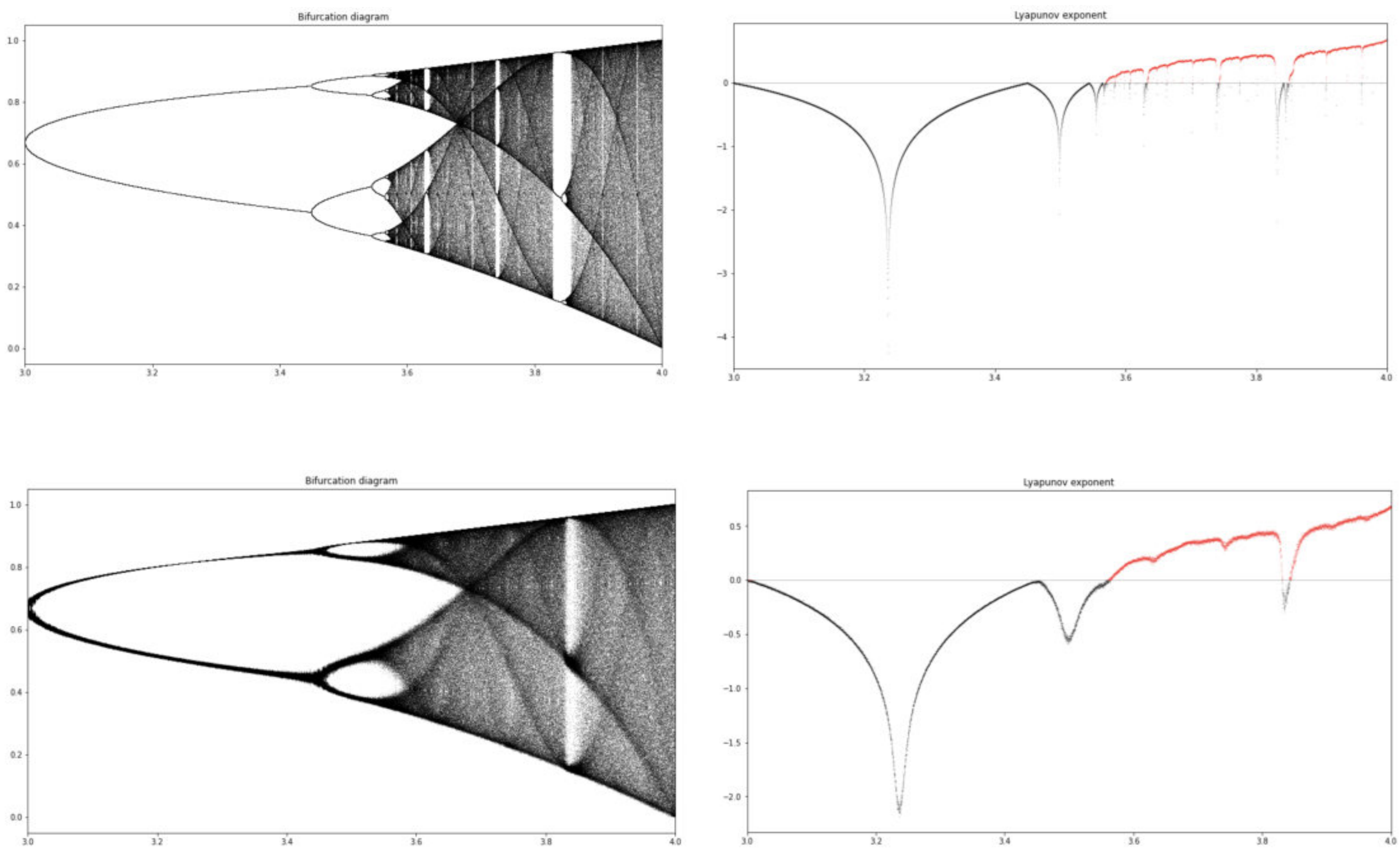

Figure 6

Logistic map: bifurcation diagram and corresponding lyapunov exponents graph

High-flow-rate-chaotic state BZ model

Traced return map:

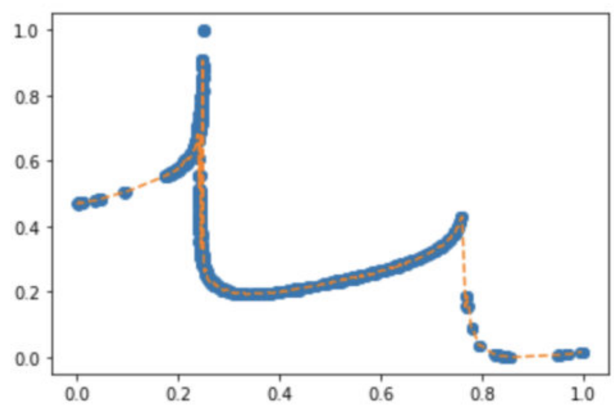

Figure 7

Belousov-Zhabotinsky Reaction, high-flow-rate-chaotic state model: data tracing

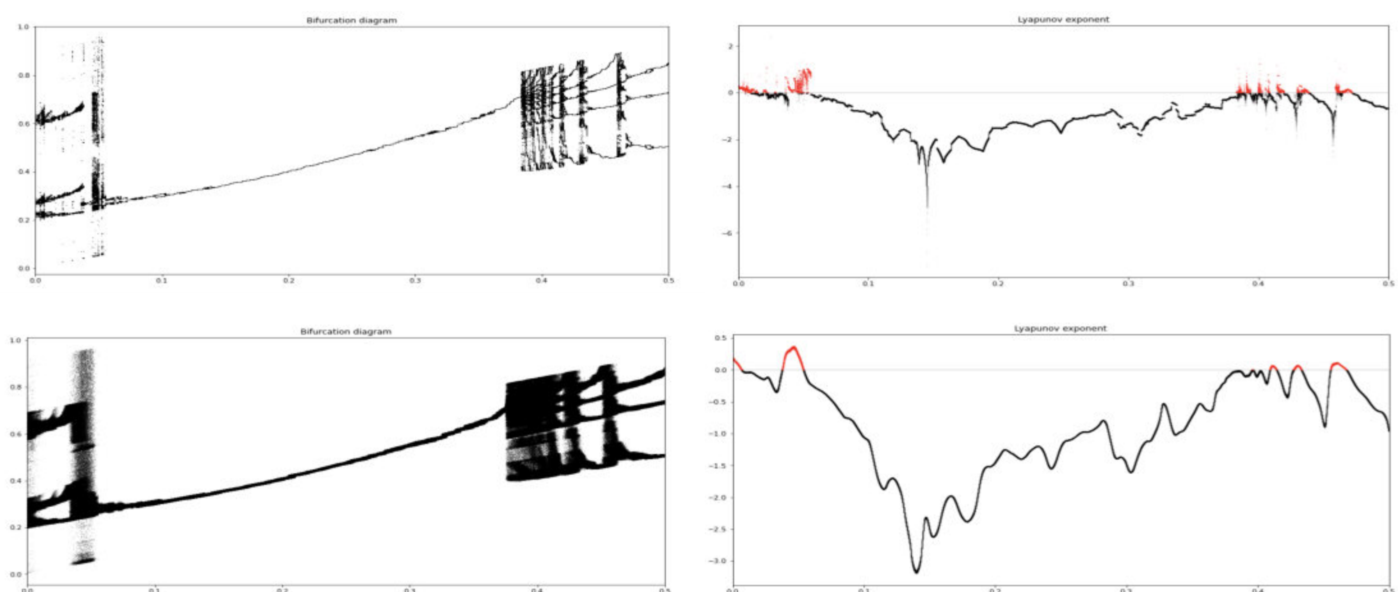

Figure 8

High-flow-rate-chaotic state model: bifurcation diagram and corresponding lyapunov exponents graph

Discussion

In the graphs generated, it was found that the Belousov Zhabotinsky Matsumoto-Tsuda model was confirmed to become less chaotic as noise increases. When enough noise is added, there is a point where the Lyapunov exponent becomes negative throughout the bifurcation diagram, which has essentially become a line, meaning that complete order has been achieved, therefore supporting the previous claim. However, the logistic map as well as the Belousov Zhabotinsky high-flow-rate chaotic state model did not display noise induced order.

In the logistic model, a significant amount of noise was added to the system, however, the Lyapunov exponent failed to become negative throughout, though the graph did smooth out a bit. The significant amount of chaotic behavior displayed in the original graph was maintained. Originally, without noise, the lyapunov exponent graph displayed vertical, indifferentiable areas. These became differentiable as noise was added. However, it generally retained its sections of positive Lyapunov exponent values and the bifurcation only became slightly “blurred” and fuzzy as seen in figure 6.

In the high-flow-rate-chaotic state BZ model, similar to the logistic model, noise induced order did not occur. The bifurcation diagram “blurred” slightly and the Lyapunov exponent curve retained its positive sections. However, it can be observed that the red region parameters, or the regions of positive Lyapunov exponent values, became narrower. Similar to the logistic map, it did also smooth out into a continuous, differentiable appearing curve when originally, when no noise perturbations were included, there were periods of vertical slope. Additionally, it was seen that for this specific model, the original, traced map has two peaks, or humps. Similarly, in the graph for the lyapunov exponent, there were two clusters of chaotic behavior observed.

These observations can be explained by looking at the original maps of these models. In the Matsumoto-Tsuda model, the region of the map where the derivative $|T'| > 1$ is very small. Most of the map has a contracting derivative $|T'| < 1$. In the logistic map as well as in the high-flow-rate chaotic state map, the expanding region is relatively large, so there is no way to avoid the chaotic regions. Especially for the logistic map, the graph lacks many flat, or relatively flatter areas, so the chaos displayed, even after adding noise, is significantly large. On the other hand, for the high-flow-rate chaotic state map, there are more flatter areas where the derivative has a smaller magnitude, possibly resulting in the more order being displayed, and some degree of decreased chaos after the addition of noise. Additionally, the “smoothing” of the graphs can be attributed to the addition of noise decreasing the degree of randomness of chaotic orbits through probability distribution.

Conclusions

It was to be expected that the Matsumoto-Tsuda model would eventually display noise induced order as noise size increased. The behavior displayed in the lyapunov exponents of the logistic map as well as the BZ reaction high-flow-rate-chaotic state map showed that the points in the graphs of the lyapunov exponent increasingly converged towards the image of a continuous, differentiable curve, despite not fully displaying noise induced order. However, the high-flow-rate chaotic state map did indeed display some noise induced order. This behavior of smoothing was also seen in the Matsumoto-Tsuda Model, where the points seemed to converge towards a continuous, differentiable curve. This is likely due to the effect noise has on decreasing the randomness of chaotic orbits. It was also confirmed that maps displaying flatter curves are more likely to experience noise induced order.

This finding could potentially be implemented into future research and applications. It is an accurate method to predict and generalize chaotic behavior as well as order. If it is discovered that this can apply to some different situations, especially in fields like health and medicine where stability is generally preferred over chaotic behavior.It has already been proven that chaos theory could lead to significant discoveries in this field in the case of heart fibrillations where it could have significant impacts. This discovery could also aid in research for other forms of induced order. In any case, the presented data has the potential to contribute to much future research. As chaos is still a relatively less studied topic, there are many potential applications that have yet to be found.

References

Doi, S. (1989). A Chaotic Map with a Flat Segment Can Produce a Noise-Induced Order. *Journal of Statistical Physics*, *55*(5/6).

Galatolo, S., Monge, M., & Nisoli, I. (2020). Existence of noise induced order, a computer aided proof. *Nonlinearity*, *33*(9), 4237-4276. https://doi.org/10.1088/1361-6544/ab86cd

Matsumoto, K., & Tsuda, I. (1983). Noise-induced order. *Journal of Statistical Physics*, *31*(1), 87-106. https://doi.org/10.1007/BF01010923

May, R. M. (1976). Simple mathematical models with very complicated dynamics. *Nature*, *261*(5560), 459-467. https://doi.org/10.1038/261459a0

Nisoli, I. (2021, September 28). *How does noise induce order?* arXiv.org. Retrieved November 8, 2021, from https://arxiv.org/abs/2003.08422v4

Peitgen, H.-O., Jürgens, H., & Saupe, D. (2007). *Chaos and fractals: New frontiers of science* (2nd ed.). Springer.

Weiss, J. N., Garfinkel, A., Karagueuzian, H. S., Qu, Z., & Chen, P.-S. (1999). Chaos and the transition to ventricular fibrillation. Circulation, 99(21), 2819-2826. https://doi.org/10.1161/01.CIR.99.21.2819

Zhang, D., Györgyi, L., & Peltier, W. R. (1993). Deterministic chaos in the Belousov–Zhabotinsky reaction: Experiments and simulations. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *3*(4), 723-745. https://doi.org/10.1063/1.165933