Local structure of fluid equilibria and the action principle

Theodore D. Drivas

ABSTRACT. We review two versions of Arnold's variational principle for stationary states of the Euler equation (extremizing the energy functional either over the orbit in the group of area preserving diffeomorphisms of a given vorticity ω_0, or of a given streamfunction ψ_0 [1, 2]). The former is relevant for the dynamics of the Euler equation, since the Euler vector field is tangent to isovorticity leaves and so energy maximizers/minimizers are Lyapunov stable. The latter is relevant to particle dynamics since all other fields on the isostream leaf have Lagrangian dynamics conjugate to those generated by ψ_0 as the Hamiltonian. We also discuss their implications for the local structure of the manifold of steady states, by analogy with a theorem in finite dimensions.

Recall the finite dimensional result (Arnold–Khesin [2, Theorem 3.3]) locally characterizing fixed points

Theorem 1. Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and consider the ODE

$$\dot{X}(t) = b(X(t)). \tag{1}$$

Suppose there are first integrals $f : \mathbb{R}^n \to \mathbb{R}^k$, e.g. the vector field b is tangent to the surfaces

$$S_{\mathsf{h}} := \{x \in \mathbb{R}^n \ : \ \mathsf{f}(x) = \mathsf{h} \in \mathbb{R}^k\}. \tag{2}$$

Suppose also that the system possesses an additional first integral $E : \mathbb{R}^d \to \mathbb{R}$, i.e.

$$b \cdot \nabla E = 0. \tag{3}$$

Let $x_0 \in \mathbb{R}^d$ be such that

- there exists a regular point[1] $h_0 \in \mathbb{R}^k$ of f such that $x_0 \in S_{\mathsf{h}_0}$, i.e. h_0 is so that the Jacobian matrix $J_{\mathsf{f}} := \nabla \mathsf{f} : \nabla \mathsf{f}^T : \mathbb{R}^n \to \mathbb{R}^{k\times k}$ is non-singular (has rank k), i.e.

$$|J_{\mathsf{f}}| := [\det J_{\mathsf{f}}]^{1/2} \neq 0 \quad \text{for all} \quad x \in S_{\mathsf{h}_0}. \tag{4}$$

- x_0 is a critical point of E restricted to the leaf S_{h_0}
- the second differential of E restricted to the leaf is a nondegenerate quadratic form

Then the following hold true

- x_0 is a fixed point $b(x_0) = 0$,
- x_0 is Lyapunov stable
- there exists ε so that for all $\mathsf{h} \in \mathbb{R}^k$ such that $|\mathsf{h} - \mathsf{h}_0| \leq \varepsilon$ the leaf S_{h} contains a fixed point of b which is a conditional nondegenerate maximum or minimum (same as x_0) of E.

As such, neighboring fixed points are stable and form a locally smooth k-dimensional submanifold.

PROOF OF THEOREM 1. Recall that for any function $g : \mathbb{R}^n \to \mathbb{R}$ and $(n-k)$–dimensional submanifold S_{h}, we may express its gradient in terms of

$$\nabla g = \mathbf{P}_x \nabla g + \mathbf{P}_x^{\perp} \nabla g, \qquad \text{where} \quad \mathbf{P}_x^{\perp} \nabla g = \sum_{i=1}^{k} \lambda_i \nabla f_i, \tag{5}$$

where $\mathbf{P}_x$ is the orthogonal projection onto $T_x S_{\mathsf{h}}$ and $\lambda = \lambda(x) : \mathbb{R}^n \to \mathbb{R}^k$ are defined by

$$\lambda(x; g) := J_f^{-1}(x)(\nabla f \cdot \nabla g)(x). \tag{6}$$

Let $\tilde{g} = g|_{S_{\mathsf{h}_0}}$ be the restriction of g to S_{h}. For $x \in S_{\mathsf{h}}$, the surface gradient $\tilde{\nabla} := \mathbf{P}_x \nabla$ on S_{h} of $\tilde{g}$ is

$$\tilde{\nabla} \tilde{g} = (\mathbf{P}_x \nabla g)|_{S_{\mathsf{h}}} \tag{7}$$

[1] By Sard's theorem, provided $f \in C^{n-k+1}$, the set of critical points (those points where the Jacobian matrix has rank $< k$) is zero measure. A regular points h_0, S_{h_0} is a locally smooth (co-dimension k) submanifold, and there is an open neighborhood of x_0 in $\mathbb{R}^d$ so that the the surfaces S_{h} form a foliation.

We remark that the discovery of Levi-Civita was that $\tilde{\nabla} := \mathbf{P}_x \nabla$ is defined uniquely by the submanifold S_{h} itself and does not depend on the enveloping space $\mathbb{R}^n$. Thus, if x_0 is a critical point of the first integral $E : \mathbb{R}^n \to \mathbb{R}$ on the leaf S_{h_0}, i.e. $\tilde{\nabla}\tilde{E}(x_0) = 0$, its gradient in $\mathbb{R}^n$ at x_0 satisfies

$$\nabla E(x_0) = \sum_{i=1}^{k} \lambda_i \nabla f_i(x_0) \tag{8}$$

for appropriate scalars $\{\lambda_i\}_{i=1}^k$ given by (6) with $g = E$. The second differential restricted to S_{h_0} is

$$\tilde{\nabla} \otimes \tilde{\nabla}\tilde{E} = (\mathbf{P}_x \nabla_x \otimes \mathbf{P}_x \nabla_x E)|_{S_{\mathsf{h}_0}} \tag{9}$$

and, according to (5), coincides with the restriction of the differential of $\nabla E - \sum_{i=1}^k \lambda_i \nabla f_i$ in $\mathbb{R}^n$, e.g.

$$\begin{aligned} \widetilde{\mathrm{Hess}}\tilde{E} &= \mathrm{Hess}E\big|_{T_x S_{\mathsf{h}_0}} - \sum_{i=1}^{k} \lambda_i \mathrm{Hess} f_i\big|_{T_x S_{\mathsf{h}_0}} - \sum_{i=1}^{k} \nabla\lambda_i \otimes \nabla f_i\big|_{T_x S_{\mathsf{h}_0}} \\ &= \mathrm{Hess}E\big|_{T_x S_{\mathsf{h}_0}} + \mathrm{II}_{x_0}(\cdot,\cdot)\cdot \nabla E\big|_{T_x S_{\mathsf{h}_0}} \end{aligned} \tag{10}$$

since $\nabla\lambda_i \otimes \nabla f_i$ is orthogonal to TS_h and where the second fundamental form of the surface

$$\mathrm{II}_x(u,v) = \mathbf{P}_x^{\perp}[\nabla_v u], \qquad \text{for} \qquad u,v \in T_x S_h \tag{11}$$

Proof that x_0 is a fixed point. Let $X(t)$ denote the solution (1) with $X(0) = x_0$. In view of (10) & (8),

$$\begin{aligned} 0 = \frac{\mathrm{d}^2}{\mathrm{d}t^2} E(X(t))\Big|_{t=0} &= \mathrm{Hess}E|_{X(t)}(\dot{X}(t),\dot{X}(t))\Big|_{t=0} + \ddot{X}(t)\cdot\nabla E(X(t))\Big|_{t=0} \\ &= b\otimes b : \mathrm{Hess}E|_{x_0} + (\nabla_b b)\cdot \nabla E|_{x_0} = \widetilde{\mathrm{Hess}}\tilde{E}(b,b)\big|_{x_0}. \end{aligned}$$

Since, by assumption $\widetilde{\mathrm{Hess}}\tilde{E}(\cdot,\cdot)$ is a non-degenerate quadratic form, we deduce that $b(x_0) = 0$ as desired.

Proof that steady states form an k-dimensional manifold indexed by the leaves. Let $\{e_1(x), ..., e_{n-k}(x)\}$ be a orthonormal basis of eigenvectors of $\widetilde{\mathrm{Hess}}\tilde{E}(\cdot,\cdot)$, so that $\widetilde{\mathrm{Hess}}\tilde{E}(e_k, e_\ell) = \lambda_\ell \delta_{k\ell}$. This basis exists because $\widetilde{\mathrm{Hess}}\tilde{E}(\cdot,\cdot)$ is a symmetric matrix. Now define for $\ell = 1, ..., n-k$, define the maps $g_\ell : \mathbb{R}^n \to \mathbb{R}$ as $g_\ell(x) = \nabla E(x)\cdot e_\ell(x)$. Notice that at x_0, $g_\ell(x_0) = 0$. We claim that in a neighbourhood of x_0, the zero set $Z_0 := \{x, g_1(x) = ... = g_{n-k}(x) = 0\}$ is a k-dimensional manifold, transverse to the leaf S_h passing through x. The claim follows if we prove that that $\mathrm{span}\{\nabla f_1(x), ..., \nabla f_k(x), \nabla g_1(x), ..., \nabla g_{n-k}(x)\} = \mathbb{R}^n$. Indeed this implies on the one hand that the ∇g_ℓ are non-zero and linearly indepedent at each point $x \in Z_0$, so by the implicit function theorem the zero set is a smooth k-dimensional manifold. Transversality follows because the tangent spaces of both level sets span $\mathbb{R}^n$. Steady states are conditional nondegenerate maximum/minimum (depending on the nature at x_0) of E since the eigenvalues of the Hessian vary continuously.

Proof that x_0 is Lyapunov stable. Abstractly this follows simply from the fact that the intersections

$$\{E = e\} \cap S_{\mathsf{h}}, \tag{12}$$

which confine the dynamics, are codimension $k+1$ topological spheres with the property that as $e \to e_* := E(x_*(\mathsf{h}))$, they contract to a point. The diameter of the intersection sets depends smoothly on $|e - e_*|$.

□

1. Variation Principle for Euler based on Vorticity

Here we are concerned with the variational principle defined with the energy

$$\inf_{\omega\in\mathcal{O}_{\omega_0}} \int_M |K[\omega]|^2 \mathrm{d}x, \qquad K[\omega] := \nabla^{\perp}\Delta^{-1}\omega \tag{13}$$

where $\mathcal{O}_{\omega_0}$ is the orbit of ω_0 in the area preserving diffeomorphism group $\mathcal{D}_\mu(M)$:

$$\mathcal{O}_{\omega_0} = \{\psi : M \to \mathbb{R} \ : \ \omega = \omega_0 \circ \varphi, \text{ for some } \varphi \in \mathcal{D}_\mu(M)\}. \tag{14}$$

The analogues of the conditions of Theorem 2 in infinite dimensions for Euler are

- ω_0 is a vorticity function so that in some function space X, there is a neighborhood around ω_0 in X so that the topology of all vorticity functions in that neighborhood are the same as that of ω_0. This is ensured, for example, if all critical points of ω_0 are Morse, or else if ω_0 has no critical points.
- the leaves defined by "first integrals" are the orbits $\mathcal{O}_\omega$ for $\omega \in B_X(\omega_0)$.
- the functional is the energy, stationary solutions are critical points on isovortical sheets $\mathcal{O}_{\omega_0}$
- non-degenerate maximum or minimum is ensured, according to Theorem 2, for critical points ω_0 satisfying $\omega_0 = F(\psi_0)$ with F satisfying either

$$-\lambda_1(M) < F' < 0 \qquad \text{or} \qquad F' > 0, \tag{15}$$

where $\lambda_1(M)$ is the smallest positive eigenvalue of the Dirichlet Laplacian $-\Delta$ on M.

Under these hypotheses, the analogue of Theorem 1 was proved in [3]. We now discuss the point about the non-degenerate quadratic form at greater length. We analyze this situation by identifying $\int_M |K[\omega]|^2 \mathrm{d}x$ for $\omega = \omega_0 \circ \varphi$ for some $\varphi \in \mathcal{D}_\mu(M)$ with

$$E_{\omega_0}[\varphi] := \frac{1}{2} \int_M |u_\varphi(x)|^2 \mathrm{d}x \tag{16}$$

where

$$u_\varphi(x) = \nabla^\perp \psi_\varphi(x), \qquad \psi_\varphi(x) := (\Delta^{-1}\omega_\varphi)(x), \qquad \omega_\varphi(x) = \omega_0(\varphi(x)). \tag{17}$$

Theorem 2. The first and second variation of $E_{\omega_0}[\varphi]$ in the direction of $\eta := \nabla^\perp h$ are

$$\delta E_{\omega_0}[\varphi]\eta\Big|_{\varphi=\mathrm{id}} = -\int_M \{\omega_0, \psi_0\} h \mathrm{d}x \tag{18}$$

$$\delta^2 E_{\omega_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} = \int_M \Big[|\Delta^{-1/2}\{h,\omega_0\}|^2 + \{h,\omega_0\}\{h,\psi_0\}\Big] \mathrm{d}x \tag{19}$$

PROOF. The variation of along $\mathcal{O}_{\omega_0}$ is $\delta\omega_\varphi|_{\varphi=\mathrm{id}} = \{h, \omega_0\}$. Note now that we may express the energy as

$$E_{\omega_0}[\varphi] = \frac{1}{2}\int_M |u_\varphi(x)|^2 \mathrm{d}x = -\frac{1}{2}\int_M \psi_\varphi \omega_\varphi \mathrm{d}x,$$

so that the first variation is explicitly

$$\delta E_{\omega_0}[\varphi]\eta\Big|_{\varphi=\mathrm{id}} = -\int_M \psi_\varphi \delta\omega_\varphi \mathrm{d}x\Big|_{\varphi=\mathrm{id}} = \int_M \psi_0\{\omega_0, h\}\mathrm{d}x = -\int_M h\{\omega_0,\psi_0\}\mathrm{d}x$$

where we used the following the Leibnitz formula (Lemma 1 with $a = \omega_0$, $b = \psi_0$, $c = h$):

$$\psi_0\{\omega_0, h\} = -\{\omega_0,\psi_0\}h + \{\omega_0, \psi_0 h\} \tag{20}$$

and that $\{\omega_0, \psi_0 h\}$ is integral zero. The second variation along $\mathcal{O}_{\omega_0}$ is $\delta^2\omega_\varphi\Big|_{\varphi=\mathrm{id}} = \{h,\{h,\omega_0\}\}$. Thus

$$\begin{aligned}\delta^2 E_{\omega_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} &= -\int_M \Big[\psi_\varphi\delta^2\omega_\varphi + \delta\psi_\varphi\delta\omega_\varphi\Big]\Big|_{\varphi=\mathrm{id}}\mathrm{d}x = -\int_M \psi_0\{h,\{h,\omega_0\}\}\mathrm{d}x + \int_M |\nabla\delta\psi_\varphi|^2\Big|_{\varphi=\mathrm{id}}\mathrm{d}x \\ &= \int_M \{\omega_0,h\}\{\psi_0,h\}\mathrm{d}x + \int_M |\Delta^{-1/2}\{h,\omega_0\}|^2\mathrm{d}x.\end{aligned}$$

□

We used parts of the following elementary lemma about the Poisson bracket

Lemma 1. *The bracket* $\{a,b\} = \nabla^\perp a \cdot \nabla b$ *is a* Poisson structure*, e.g. it satisfies*

- *bilinear:* $\{a+c, b+d\} = \{a,b\} + \{a,d\} + \{c,b\} + \{c,d\}$,

- *skew symmetric:* $\{a,b\} = -\{b,a\}$,
- *Leibnitz:* $\{a,bc\} = \{a,b\}c + b\{a,c\}$,
- *Jacobi identity:* $\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0$.

Remark 1 (Special form for $\omega_0 = F(\psi_0)$)**.** If ω_0 satisfying the condition of being an Euler steady state $\{\omega_0, \psi_0\} = 0$ has the additional property that there is a Lipschitz $F : \mathbb{R} \to \mathbb{R}$ so that $\omega = F(\psi)$, then

$$\delta^2 E_{\omega_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} = \int_M \Big[|\Delta^{-1/2}\{h,\omega_0\}|^2 + G'(\psi_0)|\{h,\omega_0\}|^2\Big]\mathrm{d}x \tag{21}$$

where $G(\omega_0) = \psi_0$, (e.g. $G = F^{-1}$) and therefore $G'(c) = \frac{1}{F'(F^{-1}(c))}$. We have $G' > 0 \iff F' > 0$. If

$$G' > 0 \quad \Longrightarrow \quad \text{then} \quad \delta^2 E_{\omega_0}[\varphi](\eta,\eta) > 0,$$

namely, ω_0 is an energy minimizer. On the other hand, since $\{h,\omega_0\}$ is mean zero, by Poincare

$$\int_M |\Delta^{-1/2}\{h,\omega_0\}|^2\mathrm{d}x \le \frac{1}{\lambda_1}\int_M |\{h,\omega_0\}|^2\mathrm{d}x$$

so that

$$\delta^2 E_{\omega_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} \le \int_M (G'(\psi_0) + \frac{1}{\lambda_1})|\{h,\omega_0\}|^2\mathrm{d}x.$$

This quadratic form is negative definite, so that ω_0 are energy maximizers, provided

$$G' + \tfrac{1}{\lambda_1} < 0 \iff -\lambda_1 < F' < 0 \qquad \Longrightarrow \qquad \delta^2 E_{\omega_0}[\varphi](\eta,\eta) < 0.$$

2. Variation Principle for Euler based on Streamfunction

Here we are concerned with the variational principle

$$\inf_{\psi\in\mathcal{O}_{\psi_0}} \frac{1}{2}\int_M |\nabla\psi|^2\mathrm{d}x \tag{22}$$

where $\mathcal{O}_{\psi_0}$ is the orbit of ψ_0 in the area preserving diffeomorphism group $\mathcal{D}_\mu(M)$:

$$\mathcal{O}_{\psi_0} = \{\psi : M \to \mathbb{R} \ : \ \psi = \psi_0\circ\varphi, \text{ for some } \varphi\in\mathcal{D}_\mu(M)\}. \tag{23}$$

We analyze this situation by identifying $\frac{1}{2}\int_M |\nabla\psi|^2\mathrm{d}x$ for $\psi = \psi_0\circ\varphi$ for some $\varphi\in\mathcal{D}_\mu(M)$ with

$$E_{\psi_0}[\varphi] := \frac{1}{2}\int_M |u_\varphi(x)|^2\mathrm{d}x \tag{24}$$

where

$$u_\varphi(x) = \nabla^\perp\psi_\varphi(x), \qquad \psi_\varphi(x) := \psi_0(\varphi(x)), \qquad \omega_\varphi(x) := \Delta\psi_\varphi(x). \tag{25}$$

Theorem 3. The first and second variation of $E_{\psi_0}[\varphi]$ in the direction of $\eta := \nabla^\perp h$ are

$$\delta E_{\psi_0}[\varphi]\eta\Big|_{\varphi=\mathrm{id}} = \int_M \{\omega_0,\psi_0\}h\mathrm{d}x, \tag{26}$$

$$\delta^2 E_{\psi_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} = \int_M \Big[|\nabla\{h,\psi_0\}|^2 + \{\omega_0,h\}\{\psi_0,h\}\Big]\mathrm{d}x. \tag{27}$$

PROOF. The first variation along $\mathcal{O}_{\psi_0}$ is $\delta\psi_\varphi|_{\varphi=\mathrm{id}} = \{h,\psi_0\}$. We compute

$$\delta E_{\psi_0}[\varphi]\eta\Big|_{\varphi=\mathrm{id}} = -\int_M \omega_\varphi\delta\psi_\varphi\mathrm{d}x\Big|_{\varphi=\mathrm{id}} = \int_M \omega_0\{\psi_0,h\}\mathrm{d}x = \int_M h\{\omega_0,\psi_0\}\mathrm{d}x.$$

The second variation along $\mathcal{O}_{\psi_0}$ is $\delta^2\psi_\varphi\Big|_{\varphi=\mathrm{id}} = \{h,\{h,\psi_0\}\}$. Thus

$$\delta^2 E_{\psi_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} = -\int_M \Big[\omega_\varphi\delta^2\psi_\varphi + \delta\psi_\varphi\delta\omega_\varphi\Big]\Big|_{\varphi=\mathrm{id}}\mathrm{d}x = -\int_M \Big[\omega_0\{h,\{h,\psi_0\}\} + |\nabla\delta\psi_\varphi|^2\Big|_{\varphi=\mathrm{id}}\Big]\mathrm{d}x$$
$$= \int_M \{\omega_0,h\}\{\psi_0,h\}\mathrm{d}x + \int_M |\nabla\{h,\psi_0\}|^2\mathrm{d}x.$$

□

Remark 2 (Special form for $\omega_0 = F(\psi_0)$)**.** If ω_0 satisfying the condition of being an Euler steady state $\{\omega_0,\psi_0\} = 0$ has the additional property that there is a Lipshitz $F : \mathbb{R} \to \mathbb{R}$ so that $\omega = F(\psi)$, then

$$\delta^2 E_{\psi_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} = \int_M \Big[|\nabla\{h,\psi_0\}|^2 + F'(\psi_0)|\{\psi_0,h\}|^2\Big]\mathrm{d}x. \tag{28}$$

In this case, since $\{h,\psi_0\}$ is mean zero, by Poincare's inequality we have

$$\delta^2 E_{\psi_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} \geq \int_M (F'(\psi_0)+\lambda_1)|\{\psi_0,h\}|^2\mathrm{d}x$$

Thus, if $F'(\psi_0) > -\lambda_1$, we have that the form is positive and energy is a minimum.

Remark 3 (Relation between the actions)**.** Note that

$$\delta^2 E_{\omega_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} = \int_M \Big[|\Delta^{-1/2}\{h,\omega_0\}|^2 + \{h,\omega_0\}\{h,\psi_0\}\Big]\mathrm{d}x$$
$$= \delta^2 E_{\psi_0}[\varphi](\eta,\eta)\Big|_{\varphi=\mathrm{id}} + \int_M \Big[|\Delta^{-1/2}\{h,\omega_0\}|^2 - |\nabla\{h,\psi_0\}|^2\Big]\mathrm{d}x \tag{29}$$

Remark 4 (Dynamical Significance)**.** The content of the variational principle (22) is the following. Consider the Lagrangian dynamics of generated by the streamfunction ψ_0 as the Hamiltonian:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t = \nabla^\perp\psi_0(\Phi_t), \qquad \Phi_0 = \mathrm{id}.$$

Being Hamiltonian, the flow Φ_t preserves the levels sets $\{\psi_0 = c\}$ of ψ_0. Thus, the dynamics are determined entirely from the period of revolution $\mu(c)$ of a particle confined to the level curves $\{\psi_0 = c\}$. This period can be expressed by the following formula

$$\mu(c) = \oint_{\{\psi_0=c\}} \frac{\mathrm{d}\ell}{|\nabla\psi|}.$$

Indeed, if we parametrize $\Gamma_c := \{\psi_0 = c\}$ by $\Phi_\cdot : [0,\mu(c)] \to \Gamma_c$, then

$$\oint_{\{\psi_0=c\}} \frac{\mathrm{d}\ell}{|\nabla\psi|} = \int_0^{\mu(c)} \frac{|\dot\Phi_t|}{|\nabla\psi(\Phi_t)|}\mathrm{d}t = \int_0^{\mu(c)} \frac{|\nabla\psi(\Phi_t)|}{|\nabla\psi(\Phi_t)|}\mathrm{d}t = \mu(c).$$

Supposing that the level sets $\{\psi_0 = c\}$ are topological circles that foliate a simply connected region with $\{\psi_0 = 0\}$ being a single point, the area $A(c)$ of the topological disk $\{\psi_0 \leq c\}$ can be expressed via the coarea formula as

$$A(c) = \int_{\{\psi_0\leq c\}} \mathrm{d}x = \int_0^c \oint_{\{\psi_0=c'\}} \frac{\mathrm{d}\ell}{|\nabla\psi|}\mathrm{d}c' = \int_0^c \mu(c')\mathrm{d}c'.$$

As such $\mu(c) = A'(c)$. Consequently, the orbit $\mathcal{O}_{\psi_0}$ consists of those Hamiltonians whose dynamics is conjugate to that of ψ_0 since area enclosed by sublevel sets are invariant as are the values of the streamfunction, so the hence the periods of revolution are also invariant. See [5, 4].

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