A nonlinear stochastic representation for non-diffusive densities

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ABSTRACT. We obtain a (nonlinear) stochastic representation for the solution of the continuity equation.

Let $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$ and suppose the density $\varrho = \varrho(x,t)$ solves the continuity equation

$$\partial_t \varrho + \nabla \cdot (\varrho u) = 0 \tag{1}$$

$$\varrho|_{t=0} = \varrho_0 \tag{2}$$

on $\Omega \times [0,T]$. It is well known that upon introducing Lagrangian trajectories,

$$\dot{X}_t(x) = u(X_t(x),t), \qquad X_0(x) = x, \tag{3}$$

the solution of (1), (2) admits the representation

$$\varrho(x,t) = \varrho_0(A_t(x)) \det(\nabla A_t(x)) \tag{4}$$

where $A_t := X_t^{-1}$ is the back-to-labels map. Our goal here is to obtain an alternate representation formula involving noisy paths. Specifically, we prove

Proposition 1. *Let $u \in C(0,T; C^2(\Omega))$ and ϱ be the unique strong solution of (1), (2) with $\varrho_0 > 0$. Then, ϱ admits the stochastic representation*

$$\varrho(x,t) = \mathbb{E}\left[\varrho_0(\tilde{A}_t(x)) \det(\nabla \tilde{A}_t(x))\right], \tag{5}$$

where $\tilde{A}_t := \tilde{X}_t^{-1}$ is the back-to-labels map and $\tilde{X}_t$ is the stochastic flow associated to the Itô SDE

$$\mathrm{d}\tilde{X}_t(a) = u(\tilde{X}_t(a),t)\mathrm{d}t + \sqrt{\frac{\mu(t)}{\varrho(\tilde{X}_t(a),t)}}\mathrm{d}\tilde{W}_t, \qquad X_0(a) = a \tag{6}$$

where $\tilde{W}_t$ is a d–dimensional Brownian motion and $\mu(t)$ is an arbitrary bounded positive function of time.

PROOF. To establish the representation (5), we first note that the forward Kolmogorov equation for the transition density $p(a,0|x,t)$ of the process $\tilde{X}_t(a)$ is

$$\partial_t p + \nabla \cdot (pu) = \frac{\mu}{2}\Delta(\varrho^{-1}p), \tag{7}$$

$$p|_{t=0} = \delta(x-a). \tag{8}$$

See, e.g. [1, 2]. Now, suppose that the initial data (8) is randomly distributed and described by a probability density function ϱ_0, i.e. the initial data for the continuity equation (1), (2). Define

$$\bar{p}(x,t) := \mathbb{E}_0[p(\tilde{a},0|x,t)] := \int_\Omega \varrho_0(a)p(a,0|x,t)\mathrm{d}a. \tag{9}$$

which represents the evolved probability density starting from the distribution ϱ_0. Averaging Eqs. (7), (8) and using linearity, we have

$$\partial_t \bar{p} + \nabla \cdot (\bar{p} u) = \frac{\mu}{2} \Delta(\varrho^{-1} \bar{p}), \tag{10}$$

$$\bar{p}|_{t=0} = \varrho_0. \tag{11}$$

This is also called the Fokker-Planck equation. Clearly $\bar{p}(x,t) \equiv \varrho(x,t)$ is a solution of (10), (11). Since u is assumed smooth and there are no vacuum states $\varrho_0 > 0$, it follows by uniqueness for both the systems (1), (2) and (10), (11) that this is only solution. Recall that, by definition, the transition probability (solution of the forward Kolmogorov equation) has representation

$$p(a, 0|x, t) = \mathbb{E}[\delta(\tilde{X}_t(a) - x)]. \tag{12}$$

Thus, we obtain the following representation for the solution of the continuity equation (1), (2) in terms of the solution of the (averaged) forward Kolmogorov equation (10), (11)

$$\begin{aligned} \varrho(x,t) &= \mathbb{E}_0[p(\tilde{a}, 0|x,t)] = \int_\Omega \varrho_0(a) \mathbb{E}[\delta(\tilde{X}_t(a) - x)] \mathrm{d}a \\ &= \mathbb{E} \int_\Omega \frac{\varrho_0(a)}{\det(\nabla X_t(a))} \bigg|_{a = A_t(a')} \delta(a' - x) \mathrm{d}a' = \mathbb{E}\left[\frac{\varrho_0(a)}{\det(\nabla \tilde{X}_t(a))} \bigg|_{a = \tilde{A}_t(x)} \right]. \end{aligned}$$

Finally, note that since $X_t(A_t(x)) = x$, we have that $\nabla X_t(A_t(x)) \nabla A_t(x) = \mathbb{I}$ and consequently

$$\det(\nabla A_t(x)) = [\det(\nabla X_t(A_t(x)))]^{-1}. \tag{13}$$

The result (5) follows. □

Remark 1. We remark that the formula (5) is highly nonlinear in the sense of McKean – (5) together with (6) constitute a fixed-point problem which could, in principle, be solved to obtain ϱ. This is, of course, much more difficult than is necessary, since (4) already provides a representation of the solution ρ.

Remark 2. An analogous argument can be employed to show

$$\varrho_0(x) = \mathbb{E}\left[\frac{\varrho(a,t)}{\det(\nabla \tilde{A}_t(a))} \bigg|_{a = \tilde{X}_t(x)} \right] = \mathbb{E}\left[\varrho(\tilde{X}_t(x), t) \det(\nabla \tilde{X}_t(x)) \right]. \tag{14}$$

To understand the determinant appearing above, we recall the following elementary Lemma

Lemma 1. *Fix smooth $u_t : [0,T] \times \Omega \mapsto \mathbb{R}^d$ and $\sigma_t : [0,T] \times \Omega \mapsto \mathbb{R}^{d \times d}$. Let $x \mapsto X_{s,t}(x)$ be the regular stochastic flow of diffeomorphisms [3] associated to the Itô SDE*

$$\mathrm{d}X_t(x) = u_t(X_t(x)) \mathrm{d}t + \sigma_t(X_t(x)) \cdot \mathrm{d}W_t \qquad X_0(x) = x. \tag{15}$$

Then the following formula for the Jacobian holds

$$\det(\nabla X_t(x)) = \exp\left(\int_0^t \left(\nabla \cdot u_t - \frac{1}{2} \partial_i \sigma_{jk} \partial_j \sigma_{ik} \right) \bigg|_{X_s(x)} \mathrm{d}s + \int_0^t \nabla \cdot \sigma_t \big|_{X_s(x)} \cdot \mathrm{d}W_s \right). \tag{16}$$

We omit the proof, which is a straightforward computation. Note that even if the velocity is divergence-free, the stochastic flow still exhibits volume changes due to the non-constant multiplicative coefficients. In (5), $\sigma_t := \sqrt{\mu(s)/\varrho(x,t)}\mathbb{I}$. Thus, $\nabla \cdot \sigma_t = -\sqrt{\frac{\mu}{2}} \frac{\nabla \varrho}{\varrho^{3/2}}$ and $\partial_i \sigma_{jk} \partial_j \sigma_{ik} = \frac{\mu}{2} \frac{|\nabla \rho|^2}{\varrho^3}$.

Remark 3. In the case where u is incompressible, Eq. (5) provides a representation for an ideal advected scalar. In particular, if θ solves

$$\partial_t \theta + u \cdot \nabla \theta = 0 \tag{17}$$

$$\theta|_{t=0} = \theta_0, \tag{18}$$

with $\theta_0 > 0$, there is a representation formula

$$\theta(x,t) = \mathbb{E}\left[\theta_0(\tilde{A}_t(x))\det(\nabla\tilde{A}_t(x))\right], \tag{19}$$

with

$$\det(\nabla\tilde{A}_t(x)) = \exp\left(-\left(\int_0^t \frac{\mu(s)}{4}\frac{|\nabla\theta|^2}{\theta^3}\bigg|_{X_{t,s}(x)} \mathrm{d}s + \int_0^t \sqrt{\frac{\mu(s)}{2}}\mathrm{d}W_s \cdot \frac{\nabla\theta}{\theta^{3/2}}\bigg|_{X_{t,s}(x)}\right)\right) \tag{20}$$

where $X_{t,s} = X_{s,t}^{-1}$ is the inverse of the forward flow.

1. "Energy dissipation" in Pressureless Compressible Navier-Stokes

We can use our new stochastic representation to explain energy dissipation in a sticky-particle model via a convexity argument. For $\mu > 0$. The particular sticky-particle model we consider is

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) = \mu\Delta u \tag{21}$$

$$\partial_t\rho + \nabla\cdot(\rho u) = 0 \tag{22}$$

which is the pressureless Navier-Stokes equation with constant dynamic viscosity. Strong solutions satisfy

$$\partial_t u + u\cdot\nabla u = \mu\rho^{-1}\Delta u, \tag{23}$$

$$\partial_t\rho + \nabla\cdot(\rho u) = 0. \tag{24}$$

It follows that u has the stochastic representation (using the trajectories (6))

$$u(x,t) = \mathbb{E}\left[u_0(\tilde{A}_t(x))\right],$$

Now consider for any convex function ψ, the convex entropy $\psi(u)$. By Jensen's inequality,

$$\psi\left(u(x,t)\right) \le \mathbb{E}\left[\psi\left(u_0(\tilde{A}_t(x))\right)\right].$$

Using the above inequality together with the representation (14), we have

$$\int_{\mathbb{R}^n}\mathrm{d}x\,\psi\left(u(x,t)\right)\rho(x,t) \le \int_{\mathbb{R}^n}\mathrm{d}x\,\mathbb{E}\left[\psi\left(u_0(\tilde{A}_t(x))\right)\right]\rho(x,t)$$

$$= \int_{\mathbb{R}^n}\mathrm{d}a\,\psi\left(u_0(a)\right)\mathbb{E}\left[\frac{\rho(x,t)}{\det(\nabla\tilde{A}_t(x))}\bigg|_{x=\tilde{X}_t(a)}\right] = \int_{\mathbb{R}^n}\mathrm{d}a\,\psi\left(u_0(a)\right)\rho_0(a).$$

This is the statement that density-weighted convex entropies are dissipated by the dynamics of (23). An easy calculus proof of this which goes as follows. Any function $\psi := \psi(u)$ evolves according to

$$\partial_t\psi(u) + u\cdot\nabla\psi(u) = \mu\rho^{-1}\psi'(u)\Delta u. \tag{25}$$

Using the continuity equation, we have

$$\partial_t(\rho\psi(u)) + \nabla\cdot\big(\rho\psi(u) - \mu\psi'(u)\nabla u\big) = -\mu\psi''(u)|\nabla u|^2. \tag{26}$$

The result follows upon integration.

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