Local pressure formulae for incompressible fluids equations

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ABSTRACT. In this note, we derive (a slight generalization of) a "local" formula for the pressure in incompressible hydrodynamics due to P. Constantin [1]. The identity, in particular, tells that C^α velocities have $C^{2\alpha}$ pressure fields.

We are concerned with the pressure field p of incompressible fluid motion. Specifically, fix $M \subset \mathbb{R}^d$ (possibly with boundary) and consider the system

$$\partial_t u + u \cdot \nabla u = -\nabla p + f \qquad \text{in } M, \tag{1}$$

$$\nabla \cdot u = 0 \qquad \text{in } M. \tag{2}$$

If $f = 0$, this is the incompressible Euler equations. Here, we consider f to be solenoidal $\nabla \cdot f = 0$ and nothing else. In particular, it may represent a body force of be solution dependent, e.g. $f = \nu \Delta u$ making the above system the incompressible Navier-Stokes equation. The role of the pressure function $p : M \times \mathbb{R} \to \mathbb{R}$ is to maintain the divergence-free constraint (2) under evolution. In particular, it satisfies the following elliptic equation

$$-\Delta p = \nabla u^t : \nabla u. \tag{3}$$

with appropriate Neumann conditions (if M has boundary) which arise from tracing equation (1) along the boundary. The following formula [1, Lemma 2] is due to Constantin:

Proposition 1. *Fix $v \in \mathbb{R}^3$. Let $x \in M$ and $0 < r < \operatorname{dist}(x, \partial M)$. Suppose $u \in C^2(M)$ and p solves (3). Then*

$$\begin{aligned} p(x,t) - \frac{1}{|S_r|}\int_{S_r(x)} p(y,t)dS(y) &= -\frac{1}{d}|u(x,t)-v|^2 \\ &+ \frac{1}{|S_r|}\int_{S_r(x)} |\hat{n}(y)\cdot(u(y,t)-v)|^2 dS(y) \\ &+ \int_0^r \frac{d\rho}{\rho^{d-2}} \int_{S_\rho(x)} \left(d|\hat{n}(y)\cdot(u(y,t)-v)|^2 - |u(y,t)-v|^2\right) dS(y) \end{aligned} \tag{4}$$

where $\hat{n}(y) = (y-x)/|y-x|$ and $|y-x| = r$.

We gather some preliminaries before proceeding to the proof. We work on the whole space for simplicity. In order to invert the Laplacian in 3 on the whole space, we require the Newton potential

$$N(y) = \begin{cases} -\frac{1}{2\pi}\ln|y| & d = 2 \\ \frac{1}{d(d-2)\alpha_d}|y|^{2-d} & d > 2 \end{cases} \tag{5}$$

where $\alpha_d = \pi^{d/2}/\Gamma(d/2+1)$ gives the volume of the unit d-ball. Assuming $f \in C^2$ and define

$$\varphi(x) = \int_{\mathbb{R}^d} f(x-y)N(y)dy, \tag{6}$$

Then φ is the solution to

$$-\Delta\varphi(x) = f(x) \tag{7}$$

for all $x \in \mathbb{R}^d$. If further $|\varphi| \to 0$ as $|x| \to \infty$ then the above representation formula yields the unique solution to the Poisson equation. Inserting $f := \partial_i\partial_j g_{ij} \in C^0$, and integrating by parts, we find

$$\varphi(x) = -\frac{\delta_{ij}}{d} g_{ij}(x) + p.v. \int_{\mathbb{R}^d} K_{ij}(y) g_{ij}(x-y)dy \tag{8}$$

where

$$K_{ij}(y) = \frac{y_i y_j - \frac{\delta_{ij}}{d}|y|^2}{\alpha_d |y|^{d+2}}, \qquad \forall y \in \mathbb{R}^d \setminus \{0\} \tag{9}$$

Note that choosing $g_{ij} = u_i u_j$ yields the standard non-local representation formula for the fluid pressure $\varphi = p$. In fact, since $\nabla \cdot u = 0$, for any constant vector v the pressure–Poisson equation can be written as

$$-\Delta p = \partial_i \partial_j ((u_i - v_i)(u_j - v_j)). \tag{10}$$

Thus, a generalized non-local formula for the pressure reads

$$p(x) = -\frac{1}{d}|u(x,t) - v|^2 + p.v. \int_{\mathbb{R}^d} K_{ij}(y)(u_i(y,t) - v_i)(u_j(y,t) - v_j)dy \tag{11}$$

Constantin's local pressure formula, which we shall prove below, reads

$$\begin{aligned} p(x,t) - \frac{1}{|S_r|}\int_{S_r(x)} p(y,t)dS(y) = &-\frac{1}{d}|u(x,t) - v|^2 \\ &+ \frac{1}{|S_r|}\int_{S_r(x)} |\hat{n}(y) \cdot (u(y,t) - v)|^2 dS(y) \\ &+ p.v. \int_{B_r(x)} K_{ij}(x-y)(u_i(y,t) - v_i)(u_j(y,t) - v_j)dy \end{aligned} \tag{12}$$

where $v \in \mathbb{R}^d$ is arbitrary and $\hat{n}(y) = (y - x)/|y - x|$ and $|y - x| = r$. This formula localizes (11) at the expense of added in a spherical pressure average and additional term and generalizes Constantin's formulae for arbitrary dimension (note $\alpha_3 = 4\pi/3$). Formula (12) in his paper makes the choice $v = 0$ whereas (14) chooses $v = u(x,t)$.

PROOF. Let $g_{ij} = (u_i - v_i)(u_j - v_j)$ and recall $-\Delta p = \partial_i \partial_j g_{ij}$. We recall Green's formula: Let $O \subset \mathbb{R}^d$ be some region and let $\psi \in C^2(O)$, then

$$\psi(x) = \int_O (-\Delta\psi)(y)N(x-y)dy + \int_{\partial O}[N(x-y)\partial_n\psi(y) - \psi(y)\partial_n N(x-y)]dS(y) \tag{13}$$

where we used that $-\Delta N(y) = \delta(y)$. Choosing $\psi = p$, the above formula becomes

$$\begin{aligned} p(x) &= \int_O \partial_i\partial_j g_{ij}(y)N(x-y)dy + \int_{\partial O}[N(x-y)\partial_n p(y) - p(y)\partial_n N(x-y)]dS(y) \\ &= \int_O \partial_i g_{ij}(y)\partial_j^x N(x-y)dy + \int_{\partial O} \hat{n}_j \partial_i g_{ij}(y)N(x-y)dS(y) + \int_{\partial O}[N(x-y)\partial_n p(y) + p(y)\partial_n^x N(x-y)]dS(y) \\ &= \int_O g_{ij}(y)\partial_i^x\partial_j^x N(x-y)dy \\ &\quad + \int_{\partial O}[\hat{n}_i g_{ij}(y)\partial_j^x N(x-y) + p(y)\partial_n^x N(x-y)dy + (\partial_n p(y) + \hat{n}_j\partial_i g_{ij}(y))N(x-y)]dS(y). \end{aligned}$$

We specify $O = B_r(x)$ so that $N(x-y)\big|_{y\in S_r(x)} = N(r)$. Since $\partial_n p(y) + \hat{n}_j\partial_i g_{ij}(y) = \hat{n}_j\partial_i(p\delta_{ij} + u_i u_j) = -\hat{n}\cdot\partial_t u$, we have

$$\int_{S_r(x)} (\partial_n p(y) + \hat{n}_j\partial_i g_{ij}(y))N(x-y)dS(y) = -N(r)\int_{B_r(x)} \nabla_y \cdot \partial_t u(y)dy = 0. \tag{14}$$

Now note

$$\partial_i N(x-y)\Big|_{y\in S_r(x)} = -\frac{1}{d\alpha_d}\frac{x-y}{|x-y|^d}\Big|_{y\in S_r(x)} = \frac{1}{|S_r|}\hat{n}(y), \qquad \partial_n N(x-y) = \frac{1}{|S_r|} \tag{15}$$

so that

$$\int_{S_r(x)} p(y)\partial_n^x N(x-y)dS(y) = \frac{1}{|S_r|}\int_{S_r(x)} p(y)dS(y) \tag{16}$$

and

$$\int_{S_r(x)} \hat{n}_i g_{ij}(y)\partial_j^x N(x-y)dy = \frac{1}{|S_r|}\int_{S_r(x)} \hat{n}_i(y)g_{ij}(y)\hat{n}_j(y)dy = \frac{1}{|S_r|}\int_{S_r(x)} |\hat{n}(y)\cdot(u(y,t)-v)|^2 dS(y). \tag{17}$$

And finally, using the formula (8), we have

$$\int_{B_r(x)} g_{ij}(y)\partial_i^x\partial_j^x N(x-y)dy = -\frac{1}{d}|u(x,t)-v|^2 + p.v.\int_{\mathbb{R}^d} K_{ij}(x-y)(u_i(y,t)-v_i)(u_j(y,t)-v_j)dy. \quad (18)$$

Thus, combining the above expressions we obtain the formula (12). ☐

Remark 1. To obtain representation formulae on the torus $\mathbb{T}^d = (-\pi,\pi]^d$, we must periodize,

$$K_{ij}^{per}(y) := \sum_{k\in\mathbb{Z}^d} K_{ij}(y-2\pi k), \qquad \forall y \in \mathbb{T}^d \setminus \{0\}. \quad (19)$$

Then (8), (12) hold as a representation formulae on the torus replacing $\mathbb{R}^d$ with $\mathbb{T}^d$ and $K_{ij}(y)$ with $K_{ij}^{per}(y)$.

Now we prove another interesting identity which allows for an alternate local formula

Proposition 2. *Fix $v \in \mathbb{R}^3$. Let $x \in M$ and $0 < r < \operatorname{dist}(x, \partial M)$. Suppose $u \in C^2(M)$ and p solves (3). Then*

$$\frac{1}{|B_r|}\int_{B_r(x)} p(y,t)dy - \frac{1}{|S_r|}\int_{S_r(x)} p(y,t)dS(y) = -\frac{1}{d}\frac{1}{|B_r|}\int_{B_r(x)} |u(y,t)-v|^2 dy + \frac{1}{|S_r|}\int_{S_r(x)} |\hat{n}(y)\cdot(u(y,t)-v)|^2 dS(y). \quad (20)$$

PROOF. To see this, note that

$$\begin{aligned}\int_{B_r(x)} (y-x)_i\partial_t u_i(y)dy &= -\int_{B_r(x)} (y-x)_i\partial_j(p\delta_{ij} + (u_i-v_i)(u_j-v_j))dy \\ &= -\int_{B_r(x)} \delta_{ij}(p\delta_{ij} + (u_i-v_i)(u_j-v_j))dy \\ &\quad + \int_{S_r(x)} n_j(y)(y-x)_i(p\delta_{ij} + (u_i-v_i)(u_j-v_j))dy \\ &= -\int_{B_r(x)} (dp + |u(y)-v|^2)dy \\ &\quad + r\int_{S_r(x)} (p(y) + |n(y)\cdot(u(y)-v)|^2)dy \end{aligned} \quad (21)$$

where we used that $n(y) = (y-x)/r$. On the other hand, for any divergence free vector field ψ, we have

$$\begin{aligned}\int_{B_r(x)} (y-x)_i\psi(y)dy &= \int_{B_r(x)} |x-y|\hat{n}_i(y)\psi(y)dy = \int_0^r \rho\left(\int_{S_\rho(x)} \hat{n}_i(y)\psi(y)dS(y)\right)d\rho \\ &= \int_0^r \rho\left(\int_{B_\rho(x)} \nabla\cdot\psi(y)dy\right)d\rho = 0.\end{aligned}$$

where we employed the spherical coarea formula. Since $\nabla\cdot\partial_t u = 0$, this result applies and we obtain the desired formula (20) by recalling that $|S_r| = d|B_r|/r$ and dividing the equation (21) through by $d|B_r|$. ☐

Remark 2. Combining (20) with (12), we obtain a different 'local' pressure formula only involving ball-averages:

$$p(x,t) - \frac{1}{|B_r|}\int_{B_r(x)} p(y,t)dy = -\frac{1}{d}\left(|u(x,t)-v|^2 - \frac{1}{|B_r|}\int_{B_r(x)} |u(y,t)-v|^2 dy\right) + p.v.\int_{B_r(x)} K_{ij}(x-y)(u_i(y,t)-v_i)(u_j(y,t)-v_j)dy. \quad (22)$$

References

[1] Constantin, Peter. Local formulae for the hydrodynamic pressure and applications. Russian Mathematical Surveys 69.3 (2014): 395.

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