Geodesics on group of measure preserving diffeomorphisms do not self-intersect

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ABSTRACT. V.I. Arnold viewed ideal incompressible fluid motion as geodesic on its configuration space of volume preserving diffeomorphisms with respect to the L^2 metric [1, 2]. When the underlying manifold M is the flat 2-torus or a simply connected planar domain with smooth boundary, Ebin [3] showed that these geodesics cannot have any transverse self-intersections: if a flow revisits its state from a prior time, then the flow is time periodic. In this note, we show that the result of Ebin [3] applies in far greater generality to essentially any setting. We prove it explicitly for for surfaces of any genus, or planar domains with any number of boundary components. The result is also true in higher dimensions, which can be seen as an application of the Hodge theorem.

Let $M \subset \mathbb{R}^2$ be a bounded planar domain with smooth boundary ∂M (possibly with multiple connected components) and exterior unit normal $\hat{n}$. The incompressible Euler equations governing the motion of a fluid which is ideal and confined to M read

$$\partial_t u + u \cdot \nabla u = -\nabla p, \qquad \text{in} \quad M, \tag{1}$$

$$\nabla \cdot u = 0, \qquad \text{in} \quad M, \tag{2}$$

$$u|_{t=0} = u_0, \qquad \text{in} \quad M, \tag{3}$$

$$u \cdot \hat{n} = 0, \qquad \text{on } \partial M. \tag{4}$$

The pressure above is determined by the incompressibility condition via

$$-\Delta p = \nabla \otimes \nabla : (u \otimes u), \tag{5}$$

with trivial projection onto harmonics (the fluid is at equilibrium without external driving forces). To understand the resulting dynamics, a useful quantity to introduce is the vorticity, a two-form related to the antisymmetric part of the velocity gradient tensor. This object is Lie transported in any dimension. Here we will work in two dimensions in which the vorticity can be identified with a scalar field $\omega := \nabla^\perp \cdot u$ with $\nabla^\perp := (-\partial_2, \partial_1)$. The the vorticity evolves according to

$$\partial_t \omega + u \cdot \nabla \omega = 0 \qquad \text{in} \quad M, \tag{6}$$

$$\omega|_{t=0} = \omega_0, \qquad \text{in} \quad M. \tag{7}$$

In fact, (6)–(7) can be closed entirely in terms of the vorticity. This is a well-known fact if the domain is simply connected (has trivial homology), or if the homology is time invariant (the flat 2-torus). For more discussion on the evolution of the harmonic part of the velocity, see [4]. With this, Ebin proved

Theorem 1 (Ebin [3]). *Let M be flat 2-torus or a simply connected planar domain with smooth boundary. Then $\mathcal{D}_\mu(M)$ has no self-intersecting geodesics. That is, if $\Phi(t)$ is a geodesic, then either $\Phi : (t_1, t_2) \to \mathcal{D}_\mu(M)$ is injective or $\Phi(t)$ is periodic.*

However, in fact, the result holds in full generality no matter the topology. We have

Theorem 2. *Let $M \subset \mathbb{R}^2$ is a bounded smooth domain. Then $\mathcal{D}_\mu(M)$ has no self-intersecting geodesics. That is, if $\Phi(t)$ is a geodesic, then either $\Phi : (t_1, t_2) \to \mathcal{D}_\mu(M)$ is injective or $\Phi(t)$ is periodic.*

This is an immediate corollary of the following lemma

Lemma 1. *Let $M \subset \mathbb{R}^2$ is a bounded smooth domain. The connected components of M are denoted $\Gamma_0, \Gamma_1, \ldots \Gamma_N$ with Γ_0 bordering the unbounded connected component of $\mathbb{R}^2 \setminus M$. Given a smooth function $\omega : M \to \mathbb{R}$ and constants γ_i, there exists a unique smooth u satisfying*

$$\nabla^\perp \cdot u = \omega, \qquad \nabla \cdot u = 0, \qquad u \cdot \hat{n}|_{\partial M} = 0 \qquad \oint_{\Gamma_i} u \cdot \mathrm{d}\ell = \gamma_i, \ \textit{for} \ i = 1, 2, \ldots, N. \tag{8}$$

Remark 1. In view of Kelvin's circulation theorem, the circulation on the connected components of the boundaries are invariant in time

$$\oint_{\Gamma_i} u(t)\cdot \mathrm{d}\ell = \oint_{\Gamma_i} u_0 \cdot \mathrm{d}\ell. \tag{9}$$

As such, recovering u from (8) is determined completely by the vorticity and the initial data of the velocity.

PROOF OF THEOREM 2. If for some times $t_1 < t_2$, we have equal vorticities $\omega(t_1) = \omega(t_2)$, the according to Theorem 2, we have also equal velocities $u(t_1) = u(t_2)$. As the flowmaps are equal at the intersection $\Phi(t_1) = \Phi(t_2)$ and so are their tangents $\dot{\Phi}(t_1) = \dot{\Phi}(t_2)$, the flowmap $\Phi(t)$ must be time periodic with period $T = t_2 - t_1$. □

Remark 2. The system (8) is equivalent to the following "modified Biot-Savart law": Given a smooth vorticity function ω on a bounded domain M, a streamfunction ψ is determined as the solution of

$$\Delta\psi = \omega \qquad \text{in } M, \tag{10}$$

$$\psi\big|_{\Gamma_0} = 0, \tag{11}$$

$$\partial_\tau \psi\big|_{\Gamma_i} = 0 \qquad \text{for } i = 1, 2, \ldots, N, \tag{12}$$

$$\oint_{\Gamma_i} \partial_n \psi \, \mathrm{d}\ell = \gamma_i \qquad \text{for } i = 1, 2, \ldots, N, \tag{13}$$

where ∂_n the outward normal derivative to the domain and ∂_τ is a tangential derivative. The velocity field is represented as $u = \nabla^\perp \psi$. Provided ω is sufficiently regular, there exists a unique smooth solution ψ on $\bar{\Omega}$. The velocity field is then completely determined by $u = \nabla^\perp\psi$. We remark that the outer circulation (along Γ_0) is not specified as it is determined from the others by the divergence theorem.

Next, we prove the same result for surfaces

Theorem 3. *Let M be a genus $\mathfrak{g}$ closed surface with area form μ. Then $\mathcal{D}_\mu(M)$ has no self-intersecting geodesics. That is, if $\Phi(t)$ is a geodesic, then either $\Phi : (t_1, t_2) \to \mathcal{D}_\mu(M)$ is injective or $\Phi(t)$ is periodic.*

This follows immediately from a similar lemma

Lemma 2. *Let M be a genus $\mathfrak{g}$ closed surface with area form μ. Denote $N = 2\mathfrak{g}$ independent homologically non-trivial loops by $\Gamma_1, \ldots, \Gamma_N$. Given a smooth function $\omega : M \to \mathbb{R}$ and constants γ_i, there exists a unique smooth divergence-free vector field u satisfying*

$$\nabla^\perp \cdot u = \omega, \qquad \nabla\cdot u = 0, \qquad \oint_{\Gamma_i} u\cdot \mathrm{d}\ell = \gamma_i, \ \textit{for} \ i = 1, 2, \ldots, N. \tag{14}$$

PROOF. By Hodge decomposition theorem, any smooth vector field v in the closed surface can be uniquely written as

$$v = \nabla f + \nabla^\perp \psi + v_H$$

with f, ψ functions and the surface and v_H an harmonic vector field ($\nabla\cdot v_H = 0$, $\nabla^\perp\cdot v_H = 0$). For what follows, it is useful to recall as well that an harmonic vector field is completely determined by its circulations along a basis of homologically non-trivial loops.

Now let ψ be the unique function satisfying $\psi = \Delta^{-1}\omega$, and let v_H be the unique harmonic vector field satisfying

$$\int_{\Gamma_i} v_H \cdot \mathrm{d}\ell = -\int_{\Gamma_i} \nabla^\perp\psi\cdot \mathrm{d}\ell + \gamma_i$$

Then the vector field $u = \nabla^\perp\psi + v_H$ satisfies (14), and by Hodge decomposition, it is the unique divergence free vector field that does so. □

Remark 3. The argument above translates, mutatis mutandis, to higher dimensional manifolds. The only difference is that the role of the vorticity function ω is now played by an exact 2-form, and one should apply Hodge decomposition to the 1-form dual to the velocity field u. More precisely, since ω is exact, let α be any primitive 1-form, $d\alpha = \omega$. Then by Hodge decomposition

$$\alpha = \delta\beta + \alpha_{\text{closed}} = \delta\beta + df + \alpha_H$$

with α_H harmonic, f a function, and β a 2-form. We see first that $\delta\beta$ (but not β) is uniquely determined by ω, since any two primitives α and α' must differ by a closed 1-form, and any closed 1-form is the sum of an exact 1-form and an harmonic one.

Now if we want the primitive α (or strictly speaking, its dual vector field) to be divergence free, we must have $df = 0$; so the only freedom left in the primitive is in the choice of the harmonic part α_H. And this is the part that is fixed by the circulations: choosing the unique α_H such that

$$\int_{\Gamma_i} \alpha_i = -\int_{\Gamma_i} \delta\beta + \gamma_i.$$

So there is a unique divergence-free 1-form with the required circulations. We set u as the dual vector field.

1. Proof of Lemma 2

Existence. We shall construct the velocity field as follows

$$u = v + \nabla^{\perp}\psi_0 + \nabla\phi \tag{14}$$

where

- ψ_0 is built to match the vorticity and have trivial harmonic projection. It is the unique solution of

$$\begin{aligned} \Delta\psi_0 &= \omega \qquad \text{in} \quad M, \\ \psi_0 &= 0 \qquad \text{on} \quad \partial M \end{aligned}$$

- v is harmonic and fixes the circulations, defined by

$$v(x) := \sum_{i=1}^{N} \Big(\gamma_i - \oint_{\Gamma_i} \partial_n\psi_0 \mathrm{d}\ell\Big) K(x - x_i) \tag{14}$$

  where x_i is any points outside M encircled by the component Γ_i, and where $K(z) := -\frac{1}{2\pi}\frac{z^{\perp}}{|z|^2}$. Note that, provided $x \neq y$, K is divergence and curl free in the variable x, so v is harmonic in M. Indeed, if C is any closed curve encircling a point y exactly once, since $K(z) = -\frac{1}{2\pi}\nabla^{\perp}\log|z|$,

$$\begin{aligned} \int_C K(\cdot, y)\cdot \mathrm{d}\ell &= -\frac{1}{2\pi}\int_C \nabla^{\perp}\log|x(C(s)) - y| \cdot \hat{n}\,\mathrm{d}s = \int_C \frac{1}{2\pi}\nabla\log|x(C(s)) - y|\cdot\hat{n}\,\mathrm{d}s \\ &= \int_{\bar{C}} \frac{1}{2\pi}\Delta_x \log|x - y| \mathrm{d}x = 1 \end{aligned}$$

  where $\bar{C}$ is the interior of C and we used $\nabla^{\perp}\cdot K = \Delta\log|z| = \delta(z)$.
- ϕ is harmonic and fixes the non-tangent boundary velocity. It is the unique solution of

$$\begin{aligned} \Delta\phi &= 0 && \text{in} \quad M, \\ \partial_n\phi &= -\hat{n}\cdot(v + \nabla^{\perp}\psi_0) && \text{on} \quad \partial M \end{aligned}$$

Note that the velocity defined in this way satisfies all the conditions in (8).

To show uniqueness, we argue that the system (8) is equivalent to the system (10)–(13).

- One direction is trivial, if $u = \nabla^{\perp}\psi$ where ψ satisfies (10)–(13), then u satisfies (8).

- For the other direction, we must show the existence of a stream function. Fix any $p \in \partial M$. For any $x \in M$, let $C_{p,x}$ be a simple curve connecting p to x. Let

$$\psi_C(x) = -\int_{C_{p,x}} u^\perp \cdot \mathrm{d}\ell \tag{14}$$

We claim this formula does not depend on the connecting curve $C_{p,x}$. Indeed, let $\tilde{C}_{p,x}$ be another connecting curve homotopic to $C_{p,x}$. Let D be the (oriented) region between $C_{p,x}$ and $\tilde{C}_{p,x}$.

$$\begin{aligned}\psi_C(x) &= -\int_{C_{p,x}} u^\perp \cdot \mathrm{d}\ell = \int_{C_{p,x}} u \cdot \hat{n}\mathrm{d}s = \int_{\tilde{C}_{p,x}} u \cdot \hat{n}\mathrm{d}s + \int_D \nabla \cdot u \,\mathrm{d}x \\ &= \int_{\tilde{C}_{p,x}} u \cdot \hat{n}\mathrm{d}s = \psi_{\tilde{C}}(x)\end{aligned}$$

Now suppose that $\tilde{C}_{p,x}$ is another connecting curve that wraps multiple times along the boundary. Then, repeating the above calculation will result in additional boundary integral terms of the type

$$\int_{\Gamma_i} u^\perp \cdot \mathrm{d}\ell = -\int_{\Gamma_i} u \cdot \hat{n}\,\mathrm{d}s = 0 \tag{13}$$

As such, the formula (1) being well defined (independent of the connecting curve), means that $u = \nabla^\perp \psi$. But then conditions (11),(12) are satisfied by the fact that u is tangent to the boundary and conditions (10) and (13) are also immediately checked.

Uniqueness. Finally we show uniqueness. Suppose their were multiple streamfunctions satisfying (10)–(13). By linearity, their difference, call it $\tilde{\psi}$, satisfies

$$\begin{aligned} \Delta\tilde{\psi} &= 0 \qquad \text{in } M, \\ \tilde{\psi}|_{\Gamma_0} &= 0, \\ \partial_\tau \tilde{\psi}|_{\Gamma_i} &= 0 \qquad \text{for } i = 1, 2, \ldots, N, \\ \oint_{\Gamma_i} \partial_n \tilde{\psi}\,\mathrm{d}\ell &= 0 \qquad \text{for } i = 1, 2, \ldots, N. \end{aligned}$$

Note that $\tilde{\psi}$ can apriori take different (constant) values, c_i, on different connected components of the boundaries. We aim to show that $\tilde{\psi} = 0$. Suppose for contradiction that $\tilde{\psi} \neq 0$ and, without loss of generality, that $\psi > 0$ inside the domain. By maximum principle, $\tilde{\psi}$ achieves its maximum and minimum on the boundary ∂M. Suppose its maximum is attained on some Γ_i. Then, by the Hopf boundary–point lemma, $\partial_n \tilde{\psi}|_{\Gamma_i} \geq 0$ But this contradicts the normal derivative being mean zero on each component of the boundary. Thus $\tilde{\psi} = 0$.

References

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